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IMMOBILIZATION OF SOLIDS AND MONDRIGA QUADRATIC FORMS

Introduction

Robot hands with three point-like fingers (as we often picture them) are unable to get a steady hold on any 3-dimensional body K. Indeed, suppose that the positions of the fingers are the points p_1, p_2, p_3 in the boundary of K, ∂K. Then, any half-line (of which there are plenty) contained in K-supporting half-spaces through p_1, p_2, p_3 defines an escape direction for K.

If we lend a fourth finger to our robot hand, it is possible to choose $p_0, p_1, p_2, p_3 \in \partial K$ so that K is unable to move by a simple translation. For example, it is sufficient to ask that each p_i has a unique K-supporting plane and that the four K-supporting half-spaces intersect in a tetrahedron. However, K will still be able to wiggle a bit unless the points are chosen with extreme care:

Namely (and generically speaking), we will prove that a *necessary condition for four points $p_0, \ldots, p_3 \in \partial K$ to immobilize K is that the normal lines to K at $p_0, \ldots, p_3$ belong to one ruling of a quadratic surface.* And furthermore, that if *K is a tetrahedron and the points lie in the interior of its faces, this condition is also sufficient.*

Immobilization questions come from robotics, especially from grasping problems [5], [6]. In [1], a curvature criterion was established to decide when three points immobilize a plane figure. This curvature criterion proves that three points suffice to immobilize any convex figure, other than a disk, whose boundary has continuous curvature, and also that three points in the interior of the three sides of a triangle immobilize it if, and only if the three normal lines are concurrent (see also [2]).

1. Statement of the Main Result

First of all, we make the "immobilization" notion, which has been used intuitively above, precise. To fix ideas, we shall restrict ourselves to convex bodies in $\mathbb{R}^3$, although the generalizations are quite clear.

Let E be the Lie Group of orientation preserving Isometries of Euclidian Space $\mathbb{R}^3$. Given two sets $X, Y \subset \mathbb{R}^3$, define the $\mathcal{E}$-*motions* of X in Y to be

$$\mathcal{E}(X,Y) = \{g \in \mathcal{E} \mid g(X) \subset Y\},$$

considered as a subspace of $\mathcal{E}$.

Let $K \subset \mathbb{R}^3$ be a compact convex body. Denote by Int K the interior of K, (which is non-empty); and by $\mathbb{O}K$ its "outside", that is, $\mathbb{O}K = \mathbb{R}-$ Int K, so that $K \cap \mathbb{O}K = \partial K$.

Definition. Given $P \subset \mathbb{O}K$, we say that P *immobilizes* K if the identity map id $\in \mathcal{E}$ is an isolated point component of $\mathcal{E}(P, \mathbb{O}K)$.

Our main interest is the case when P consists of a finite set of points. In the introduction we talked about moving K away from P. But clearly it is equivalent, via the inverse map of $\mathscr{C}$, to think of moving P outside of K. And if this is impossible to do continuously, we can say, by intuition, that K is immobilized by P. Observe that if K has a non-discrete group of symmetries then it is not immobilized by anything at all.

Lemma 1. *Let K be a convex body, and let $p_1, \ldots, p_n$ be points in ∂K. Suppose that there exist K-supporting half-spaces H_i through p_i, $i = 1, \ldots, n$, such that $H = \cap_1^n H_i$ is not compact, then $p_1, \ldots, p_n$ dot not immobilize K.*

Proof. We have that for each $i = 1, \ldots, n$, H_i, is a closed half-space which contains K, and with p_i, in its boundary plane, so that $K \subset \cap_1^n H_i$. If H is not compact, we can clearly find a direction $\lambda \in \mathbb{R}^3$, $\lambda \neq 0$, such that $(H + t\lambda) \subset H$ for all $t \geq 0$. But then $p_i - t\lambda \in \mathbb{C}H \subset \mathbb{C}K$ for all $t \geq 0$. Therefore, the translation by $-t\lambda$, $t \geq 0$ is in $\mathscr{C}, \mathbb{C}K)$ and the indentity $(t = 0)$ is not an isolated point.

This simple result leads naturally to study the first non-trivial case which is the immobilization of a tetrahedron by four points in the interior of its faces. This will take up the main body of the paper, and only in the last section shall we come back to the general case of a convex body.

To state our main result another definition is needed.

Definition. Four non coplanar lines in $\mathbb{R}^3$ are said to be *semiconcurrent* if whenever another line is non-skew (i.e., concurrent or parallel) to three of them, it is also non-skew to the fourth one.

Theorem 1. Let *T be a tetrahedron, and let $p_0, \ldots, p_3$ be interior points in its four faces. Then, $p_0, \ldots, p_3$ immobilize T if and only if the normal lines to T at $p_0, \ldots, p_3$ are semiconcurrent.*

A version of this theorem, with an algebraic condition which we call quadratic dependence, is split in two in Section 2. The first half is proven using the energy function associated to the system, and an alternative topological proof is outlined. To prove the second half, one is led to study a quadratic form found in Section 3. This study leads to the definition of mondriga matrices and to a conjecture about their spectral radius, which is proven in our case of interest. In Section 4 we prove the equivalence of semiconcurrence and quadric dependence using quadric surfaces. Finally, in Section 5, we return briefly to the general case of a convex body.

2. The Energy Function

Throughout this section, T will be a fixed tetrahedron in $\mathbb{R}^3$, and $p_0, \ldots, p_3$ will be interior points on its four faces. Let $P = \{p_0, \ldots, p_3\}$.

Let $n_0, \ldots, n_3$, respectively, be outward normals to the faces of T, so that the four lines that Theorem 1 talks about are precisely $\ell_i = \{p_i + tn_i \mid t \in \mathbb{R}\}$. Since T is a tetrahedron we have that *any three of the n_i 's are linearly independent.* And furthermore, since they point out of T, we have that for some strictly positive $\alpha_0, \ldots, \alpha_3$, $\Sigma_0^3 \alpha_i n_i = 0$. We might as well include these positive factors in our choice of outward normals and assume, with no loss of generality, that

$$\sum_{i=0}^{3} n_i = 0 \tag{1}$$

This fixes $n_0, \ldots, n_3$ up to a common positive factor, and to pin them down precisely, we can ask $\Sigma_0^3 \, |n_i| = 1$. Now T is described by

$$T = \{x \in \mathbb{R}^3 \mid (x - p_i) \cdot n_i \leq 0\}\}, \tag{2}$$

where, by convention, *subindices run from* 0 *to* 3 *unless otherwise specified.* And our assumption that the points are interior to their face can now be given as twelve inequalities:

$$(p_i - p_j) \cdot n_j < 0, \quad i \neq j. \tag{3}$$

Let us define the extended energy function $\overline{E} : \mathscr{E} \to \mathbb{R}$ by

$$\overline{E}(g) = \sum_{i=0}^{3} (g(p_i) - p_i) \cdot n_i.$$

Lemma 2. *The extended energy function is invariant under translations.*

Proof. Indeed, for any $\lambda \in \mathbb{R}^3$ and using (1)

$$\overline{E}(g + \lambda) = \sum_{i=0}^{3} ((g(p_i) + \lambda) - p_i) \cdot n_i$$

$$= \sum_{i=0}^{3} (g(p_i) - p_i) \cdot n_i + \lambda \cdot \sum_{i=0}^{3} n_i = \overline{E}(g)$$

This motivates the definition of the (plain) *energy* functions as the map $E : \mathrm{S0}(3) \to \mathbb{R}$ given by

$$E(g) = \sum_{i=0}^{3} (g(p_i) - p_i) \cdot n_i \tag{4}$$

Where recall that SO(3) is the subgroup of $\mathscr{E}$ that fixes the origins, so that E is simply the restriction of $\overline{E}$ to SO(3).

Proposition 1. *The points $p_0, \ldots, p_3$ immobilize T if and only if the energy function E has an isolated maximum at* id $\in$ SO (3).

Proof. (Of the "if" side.)

Since every p_i lies in the interior of its face, see (3), then the following set is clearly an open neighborhood of id in $\mathscr{E}$

$$U = \{g \in \mathscr{E} \mid (g(p_i) - p_j) \cdot n_j < 0, \text{ for all } i \neq j\}$$

Claim 1: If $g \in U \cap \mathscr{E}(P, \complement T)$ then $\overline{E}(g) \geq 0$.

If $g \in U$ is also in $\mathscr{E}(P, \complement T)$, we must have that $(g(p_i) - p_i) \cdot n_i \geq 0$ (otherwise $g(p_i)$ would be in Int T, see (2), by the definition of U). Thus, $\overline{E}(g)$ is a non-negative sum, thereby proving the claim.

Claim 2: If $g \in U \cap \mathscr{E}(P, \complement T)$ is a translation then $g =$ id.

Suppose $g \in U$ is of the form $g(x) = x + \lambda$, with $\lambda \in \mathbb{R}^3$ and $\lambda \neq 0$. Since any three of the n_i 's are linearly independent, then $\lambda \cdot n_i \neq 0$ for some i. Then, since $\Sigma_0^3 \lambda \cdot n_i = \lambda \cdot \Sigma_0^3 n_i = 0$, we also have that for some i, which we now fix, $\lambda \cdot n_i < 0$. But then, $(g(p_i) - p_j) \cdot n_j < 0$ for all j which implies that $g(p_i) \in$ Int T, and therefore that $g \notin \mathscr{E}(P, \complement T)$, which proves the claim.

Suppose now that id is an isolated maximum of E. Take $g \in \mathscr{E}(P, \complement T)$. We must prove that if g is close enough to id, then $g =$ id.

Let V be an open neighborhood of id in SO(3), for which $E < 0$ except at id. We may assume that $V \subset U$.

For some $\lambda \in \mathbb{R}^3$, namely $\lambda = g(0)$, we have that $(g - \lambda) \in$ SO (3). Then, taking "close enough" to mean that $(g - \lambda) \in V$, Claim 1 and the hypothesis implies that $(g - \lambda) =$ id. From this, we conclude that g is the translation by λ. So that, adding to "close enough" that translation by $\lambda = g(0)$ is also in U, Claim 2 completes the proof.

Lemma 3. *For each $g \in \mathscr{E}$, there exists a unique $\lambda_g \in \mathbb{R}^3$, depending continuously on g, for which:*

$$((g(p_i) + \lambda_g) - p_i) \cdot n_i = 0, \text{ for } i = 1, 2, 3.$$

Proof. Since n_1, n_2, n_3 are linearly independent, the lemma follows because the linear system

$$\lambda \cdot n_i = (p_i - g(p_i)) \cdot n_i \quad i = 1, 2, 3$$

has a unique solution for λ, which clearly depends continuously on g.

Proof. (Completion of Proposition 1).

Suppose $W \subset U$ is an open neighborhood of id in $\mathscr{E}$, for $W \cap \mathscr{E}(P, \mathbb{C}\, T) = \{\mathrm{id}\}$. Observe that, by the continuity of λ_g in Lemma 3, we have that

$$W' = \{ g \in W \mid (g + \lambda_g) \in W \}$$

is still an open neighborhood of id $\in \mathscr{E}$.

Now, pick any $g \in W' \cap \mathrm{SO}(3)$, $g \neq \mathrm{id}$. Since $g + \lambda_g \in \mathscr{E}(P, \mathbb{C}\, T)$, we must have that $(g(p_i) + \lambda_g) \in \mathrm{Int}\, T$ for some $i \in \{0, \ldots, 3\}$. But since $(g + \lambda_g)$ leaves all but p_0 in ∂T (by Lemma 3), we must have that $(g(p_0) + \lambda_g) \in \mathrm{Int}\, T$. This implies in particular that $((g(p_0) + \lambda_g) - p_0) \cdot n_0 < 0$. Therefore, using Lemmas 2 and 3, and the definition of E, we obtain

$$E(g) = \overline{E}(g + \lambda_g) = ((g(p_0) + \lambda_g) - p_0) \cdot n_0 < 0 .$$

Which proves that id is an isolated maximum of E.

Now, we find an explicit expression for the energy function E. It is classically known that SO (3) can be locally parametrized by $\mathbb{R}^3$, assigning to each vector the positive rotation along the oriented axis it defines by an angle proportional to its magnitude. It will be easier to work this out in polar coordinates.

Given $\upsilon \in S^2$ (that is, $\upsilon \in \mathbb{R}$ with $|\upsilon| = 1$), it is easy to see that the rotation along υ by an angle t is the map $g_{\upsilon,t} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$g_{\upsilon,t}(x) = \cos t\,(x - (\upsilon \cdot x)\,\upsilon) + \sin t\,(\upsilon \times x) + (\upsilon \cdot x)\,\upsilon \tag{5}$$

where $\upsilon \times x$ is the standard Cross Product. Observe also that this definition extends to non-positive t, $(g_{\upsilon,-t} = g_{-\upsilon,t})$, and that $g_{\upsilon,0} = \mathrm{id}$ for any υ. With (υ, t) as polar coordinates of $\mathbb{R}^3$, this is the standard local diffeomorphism (for $t < \pi$) into an open neighborhood of id in SO (3). Now, substituting (5) in the energy function (4), and regrouping appropriately we obtain

$$E(\upsilon, t) \overset{\text{def}}{=} E(g_{\upsilon, t})$$

$$= \sum_{i=0}^{3} [(1 - \cos t)((\upsilon \cdot p_i)(\upsilon \cdot n_i) - (p_i \cdot n_i)) + \sin t\,(\upsilon \times p_i) \cdot n_i]\,.$$

Define a quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$, by

$$Q(\upsilon) = \sum_{i=0}^{3} (\upsilon \cdot p_i)(\upsilon \cdot n_i), \tag{6}$$

so that, using that $(\upsilon \times p) \cdot n = (p \times n) \cdot \upsilon$, we can write:

$$E(\upsilon, t) = (1 - \cos t)\,(Q(\upsilon) - \sum_{i=0}^{3} p_i \cdot n_i) + \sin t \left(\sum_{i=0}^{3} p_i \times n_i\right) \cdot \upsilon \tag{7}$$

Theorem 2. *If $\Sigma_0^3\, p_i \times n_i \neq 0$ then $p_0, \ldots, p_3$ do not immobilize T.*

Proof. Fix any $\upsilon \in S^2$ such that $(\Sigma_0^3\, p_i \times n_i) \cdot \upsilon > 0$ (we have a whole halfsphere of them). The map $\alpha(t) = E(\upsilon, t)$ has a positive derivate at 0. Indeed differentiating (7) with respect to t, we obtain $\alpha'(0) = (\Sigma_0^3\, p_i \times n_i) \cdot \upsilon > 0$. Then $\alpha(t) > 0$ for small $t > 0$, and $\alpha(t)$ does not have a maximum at $t = 0$. Therefore, id is not a maximum of E, and the theorem follows from Proposition 1.

Observe that the proof tells us a little more. If $(\Sigma_0^3\, p_i \times n_i) \cdot \upsilon > 0$ then, correcting with suitable translations as in Lemma 3, small positive rotations along υ, leave $p_0, \ldots, p_3$ outside of T.

Theorem 3. *If $\Sigma_0^3\, p_i \times n_i = 0$ then $p_0, \ldots, p_3$ immobilize T.*

Proof. In this case, equation (7) collapses to the first summand. Its first factor $(1 - \cos t)$ is positive for small t except at $t = 0$. Thus, E has an isolated maximum at id if and only if $(Q(\upsilon) - \Sigma_0^3\, p_i \cdot n_i)$ is strictly negative for all $\upsilon \in S^2$. This is proven in Theorem 4 at the end of Section 3, which is dedicated to the study of the quadratic form Q. Therefore, from Proposition 1 and Theorem 4, this Theorem follows.

Remark. The hypothesis of the previous two Theorems depend only on the normal lines $\ell_i = \{p_i + tn_i \mid t \in \mathbb{R}\}$, and not on their parametrization.

Indeed, let us call four lines $\ell_0, \ldots, \ell_3$ *directionally independent* if any three of their directional vectors are linearly independent, (as in our case of interest). In this case, we can clearly find directional vectors $n_0, \dots, n_3$ such that $\Sigma_0^3 n_i = 0$, and two choices of such differ by a common non-zero constant factor. So that $\Sigma_0^3 p_i \times n_i$ being null or not is independent of the choice. It is also independent of the points p_i because $(p_i + tn_i) \times n_i = p_i \times n_i$. This proves the following be consistent.

Definition. Four directionally independent lines $\ell_0, \dots, \ell_3$ are said to be *quadratically independent*, if whenever we choose directional vectors $n_0, \dots, n_3$ such that $\Sigma_0^3 n_i = 0$, then $\Sigma_0^3 p_i \times n_i \neq 0$ for $p_i \in \ell_i$. Otherwise they are *quadratically dependent*.

2.1. Alternative Topological Approach

We will briefly outline an alternative proof of Theorem 2, which gives some insight into the immobilization problem.

Let Π_i be plane through p_i normal n_i, (one of the extended faces of T). Consider the manifold $M_i = \mathscr{E}(p_i \Pi_i) \subset \mathscr{E}$. Since $\mathscr{E}$ is 6-dimensional, and M_i is defined by the single equation $g(p_i) \cdot n_i = p_i \cdot n_i$, then M_i is a 5-dimensional manifold passing through id. It divides $\mathscr{E}$ into those motions that send p_i outwards of T, and the ones which (at least close to id) send p_i to the interior of T. Taking the natural coordinates in the Lie Algebra of $\mathscr{E}$, one can compute that a normal vector to M_i at id is precisely $(p_i \times n_i, n_i) \in \mathbb{R}^3 \times \mathbb{R}^3$, where the first factor is tangent to pure rotations (SO (3)) and the second to translations.

Now, our assumptions on the n_i 's easily imply that the normal vectors to $M_0, \dots, M_3$ at id are linearly independent if and only if $\Sigma_0^3 p_i \times n_i \neq 0$. Thus, in this case, we have a transversal intersection and may conclude that $\cap_0^3 M_i$ is a 2-dimensional smooth manifold in a neighborhood of the identity. Therefore we have found a 2-manifold of motions which keep each point p_i in its corresponding face, and T is not immobilized.

From this point of view, it is remarkable that when the intersection is not transversal it collapses to dimension 0, as we still have to prove.

3. The Quadratic Form $Q(\upsilon)$

In this section we shall consider a system of four pointed directionally independent lines $(p_0 \in \ell_0)$, $(p_1 \in \ell_1)$, $(p_2 \in \ell_2)$, $(p_3 \in \ell_3)$ in $\mathbb{R}^3$.

To such a system, as in (6) above, we associate the quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$Q(\upsilon) = \sum_0^3 (p_i \cdot \upsilon)(n_i \cdot \upsilon), \qquad (8)$$

where n_i is a directional vector of ℓ_i satisfying $\Sigma_0^3 n_i = 0$.

The quadratic form Q (υ) is defined up to a non-zero constant factor because so is the set $n_0, \ldots, n_3$. But it can be reduced to a sign ambiguity by requiring $\Sigma_0^3 \mid n_i \mid = 1$. And furthermore, if the system is *non-degenerate* (i.e., if the planes Π_i through p_i, orthogonal to ℓ_i, $i = 0, \ldots, 3$, define a non-degenerate tetrahedron T), the directional vectors n_i, can be chosen to point out of T, so that Q depends only on the systems of pointed lines. When the system is degenerate, that is, when the planes Π_i meet in a point, the sign ambiguity cannot be removed.

Note that the quadratic form Q (υ) only depends on the pointed lines and not on where we put the origin (if we replace p_i by $p_i + q$, Q remains invariant).

The purpose of this section is to prove that if the lines $\ell_0, \ldots, \ell_3$ are quadratically dependent, then:

a) the quadratic form Q (υ)has zero as an eigenvalue if and only if the four points $p_0, \ldots, p_3$ lie on a plane, and

b) if the system is non-degenerate and the points $p_0, \ldots, p_3$ lie on the interior of the faces of the tetrahedron T, then the spectral radius of Q (υ)is smaller than the trace of Q (υ).

Lemma 4. *If the lines $\ell_0, \ldots, \ell_3$ are quadratically dependent, then the eigenvalues of Q (υ) correspond to the eigenvalues of the matrix*

$$B = (b_{ij}),\ b_{ij} = (p_i - p_0) \cdot n_j\ ;\ i, j = 1, 2, 3.$$

Proof. Let A be the matrix associated to Q (υ). To compute A, we may assume that p_0 is the origin. We start by proving that:

$$A = N^T \cdot P \tag{9}$$

where $n_i = (\eta_i, \upsilon_i, \mu_i)$, $p_i = (x_i, y_i, z_i)$, and

$$N = \begin{pmatrix} \eta_1 & \nu_1 & \mu_1 \\ \eta_2 & \nu_2 & \mu_2 \\ \eta_3 & \nu_3 & \mu_3 \end{pmatrix} \quad ; \quad P = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}.$$

First note that

$$p_i \times n_i = (\mu_i\, y_i - \nu_i\, z_i,\ \eta_i\, z_i - \mu_i\, x_i,\ \nu_i\, x_i - \eta_i\, y_i)$$

and since $\Sigma_0^3\, p_i \times n_i = 0$, then we obtain the following equalities

$$\sum_1^3 \mu_i y_i = \sum_1^3 \nu_i z_i ,$$

$$\sum_1^3 \eta_i z_i = \sum_1^3 \mu_i x_i , \tag{10}$$

$$\sum_1^3 \nu_i x_i = \sum_1^3 \eta_i y_i .$$

On the other hand, with $\upsilon = (x, y, z)$, we have

$$Q(\upsilon) = \sum_0^3 (p_i \cdot \upsilon)(n_i \cdot \upsilon) = \sum_1^3 (\eta_i x + \nu_i y + \mu_i z)(x_i x + y_i y + z_i z) =$$

$$\left(\sum_1^3 \eta_i x_i\right) x^2 + \left(\sum_1^3 \nu_i y_i\right) y^2 + \left(\sum_1^3 \mu_i z_i\right) z^2 +$$

$$\left(\sum_1^3 \eta_i y_i + \sum_1^3 \nu_i x_i\right) xy + \left(\sum_1^3 \eta_i z_i + \sum_1^3 \mu_i x_i\right) xz + \left(\sum_1^3 \nu_i z_i + \sum_1^3 \mu_i y_i\right) yz.$$

Thus, using (10), we obtain that

$$A = \begin{pmatrix} \Sigma_1^3 \eta_i x_i & \Sigma_1^3 \eta_i y_i & \Sigma_1^3 \eta_i z_i \\ \Sigma_1^3 \nu_i x_i & \Sigma_1^3 \nu_i y_i & \Sigma_1^3 \nu_i z_i \\ \Sigma_1^3 \mu_i x_i & \Sigma_1^3 \mu_i y_i & \Sigma_1^3 \mu_i z_i \end{pmatrix}$$

Therefore,

$$A = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \nu_1 & \nu_2 & \nu_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = N^T \cdot P.$$

Now, it is easy to see that the characteristic polynomial of $A = N^T \cdot P$ is equal to the characteristic polynomial of $B = P \cdot N^T$. Therefore they have the same eigenvalues.

We are now in a position to prove (a). Since the lines $\ell_0, \ldots, \ell_3$ are directionally independent, then the vectors n_1, n_2, n_3 are linearly independent and hence $\det N \neq 0$. Therefore, by the proof of Lemma 4, $\det A = 0$ if and only if the vectors $p_i - p_0$ lie on a plane.

3.1. Mondriga Matrices

From our geometric motivations, and in view of the preceding lemma, the following class of matrices arises naturally (compare with the proof of Theorem 4 below).

Definition. An $n \times n$ matrix $A = (a_{i,j})$ is called *mondriga* if it satisfies the following three conditions:

i) $a_{ii} > 0$, for every $1 \leq i \leq n$,
ii) $a_{ij} < a_{ii}$, for every $1 \leq i, j \leq n$ and $i \neq j$,
iii) $\Sigma_{i=1}^{n} a_{ij} > 0$, for every $1 \leq j \leq n$.

Furthermore, $A = (a_{i,j})$ is called *semimondriga* if its satisfies properties (i), (ii) and (iii) with inequalities instead of strict inequalities.

Conjecture 1. *Let A be an $n \times n$ mondriga matrix. Then*

$$\rho(A) < tr(A)$$

where the spectral radius, $\rho(A)$, of A is the maximum of the norms of all the eigenvalues of A and $tr(A)$ is the trace of A. Furthermore, $\rho(A) \leq tr(A)$, when A is semimondriga.

If $a_{ij} \geq 0$, for every $1 \leq i, j \leq n$, then the conjecture is true because, by the Gersgorin Theorem [see Horn and Johnson, 1985 (6.1.1)], given any eigenvalue λ of A there is $1 \leq i \leq n$ such that $a_{ii} - (\Sigma_{j \neq i} |a_{ij}|) \leq |\lambda| \leq a_{ii} + (\Sigma_{j \neq i} |a_{ij}|)$. And hence, because of (ii), $|\lambda| < tr(A)$ (or $|\lambda| \leq tr(A)$, when A is semimondriga.

For $n = 2$, the Conjecture is easily seen to be true. Next, we will prove it for $n = 3$ and when all the eigenvalues of A are real.

Lemma 5. *Let $A = (a_{ij})$ be a mondriga 3×3 matrix with real eigenvalues.*

Then

$$\rho(A) < tr(A)$$

Furthermore, $\rho(A) \leq tr(A)$, when A is semimondriga.

Proof. Suppose A is mondriga. Let $t = a_{11} + a_{22} + a_{33}$ be the trace of A and let $B = tI - A$, where I is the identity matrix. It will be enough to prove that the eigenvalues of B are greater than equal to zero. For that purpose, let $p(\lambda)$ be the characteristic polynomial of B. Since the leading coefficient of $p(\lambda)$ is -1, it will be sufficient to show that $p(0) > 0$, $p'(0) < 0$ and $p''(0) > 0$.

Evaluating $p(\lambda)$, (and in reverse order as above), we have to prove the following three inequalities (where $b_i = t - a_{ii}$),

$$0 < a_{11} + a_{22} + a_{33}, \tag{11}$$

$$a_{12}a_{21} + a_{23}a_{32} + a_{31}a_{13} < b_1 b_2 + b_2 b_3 + b_3 b_1, \tag{12}$$

$$0 < b_1 b_2 b_3 - a_{12}a_{21}b_3 - a_{23}a_{32}b_1 - a_{31}a_{13}b_2 + a_{12}a_{23}a_{31} + a_{21}a_{13}a_{32}, \tag{13}$$

when

$$0 < a_{11},\ a_{22},\ a_{33} \tag{14}$$

$$\begin{aligned} a_{12}, a_{13} &< a_{11} \\ a_{21}, a_{23} &< a_{22} \\ a_{31}, a_{32} &< a_{33} \end{aligned} \tag{15}$$

$$\begin{aligned} 0 &< a_{11} + a_{21} + a_{31} \\ 0 &< a_{12} + a_{22} + a_{32} \\ 0 &< a_{13} + a_{23} + a_{33} \end{aligned} \tag{16}$$

Inequality (11) follows immediately from (14).

By (15) and (16), $-b_1 = -(a_{22} + a_{33}) < a_{12} < a_{11}$, and $-b_2 = -(a_{33} + a_{11}) < a_{21} < a_{22}$. Then using (14) and that $a_{11}a_{22} < b_1 b_2$, we may conclude that $a_{12}a_{21} < b_1 b_2$. Similarly, $a_{23}a_{32} < b_2 b_3$ and $a_{31}a_{33} < b_3 b_1$. Therefore (12) follows.

The proof ends expressing the right hand side of (13) as a sum of positive terms:

$$\begin{aligned} &[(a_{22} - a_{21})(a_{12} + a_{22} + a_{32}) + (a_{33} - a_{31})(a_{13} + a_{23} + a_{33})]\, a_{11} + \\ &+ [(a_{33} - a_{32})(a_{13} + a_{23} + a_{33}) + (a_{11} - a_{12})(a_{11} + a_{21} + a_{31})]\, a_{22} + \\ &+ [(a_{11} - a_{13})(a_{11} + a_{21} + a_{31}) + (a_{22} - a_{23})(a_{12} + a_{22} + a_{32})]\, a_{33} + \\ &+ (a_{22} - a_{21})(a_{33} - a_{32})(a_{11} - a_{13}) + (a_{33} - a_{31})(a_{11} - a_{12})(a_{22} - a_{23}) \end{aligned}$$

We must thank Eduardo Dueñas for this factorization. For semimondriga matrices the proof is completely analogous.

The following, used in Section 2 prove Theorem 3, restates (b).

Theorem 4. *Let T be a tetrahedron; $p_0, \ldots, p_3$ four points in the interior of each face of T; $n_0, \ldots, n_3$, respectively, outer normal vectors to these faces, such that $\Sigma_0^3 n_i = 0$, and ℓ_i the line through the point p_i parallel to n_i, $i = 0, \ldots, 3$. Suppose that $\ell_0, \ldots, \ell_3$ are quadratically dependent. Then, for every unit vector $\upsilon \in S^2$*

$$Q(\upsilon) = \sum_{i=0}^{3} (p_i \cdot \upsilon)(n_i \cdot \upsilon) < \sum_{i=0}^{3} p_i \cdot n_i$$

Proof. We may assume that $p_0 = 0$. By Lemma 4, the eigenvalues of the quadratic form $Q(\upsilon)$ are the eigenvalues of the matrix $B = (b_{i,j})$, where $b_{ij} = p_i \cdot n_j$, for every $1 \leq i\ j \leq 3$. Since the points p_i are in the interior of their corresponding faces, we have that $p_i \cdot n_j < p_j \cdot n_j$ for every $0 \leq i, j \leq 3$, $i \neq j$. These twelve inequalities, together with the fact that $p_0 = 0$ and $n_0 = n_0 = -n_1 - n_2 - n_3$ are precisely the conditions for B to be a mondriga matrix. Furthermore, by Lemma 4, B has only real eigenvalues, corresponding to those of $Q(\upsilon)$. Therefore, Lemma 5 implies that the three eigenvalues of $Q(\upsilon)$ are smaller than the trace of B, namely $\Sigma_0^3 p_i \cdot n_i$, which is also the trace of $Q(\upsilon)$. Consequently, since the maximum of Q in S^2 is obtained at an eigenvector and is an eigenvalue, we have that for every $\upsilon \in S^2$, $Q(\upsilon) < \Sigma_0^3 p_i \cdot n_i$

4. Quadric Surfaces and Semiconcurrence

In this section we give the geometric interpretation of quadratic dependence.

Proposition 2. *Let $\ell_0, \ldots, \ell_3$ be four directionally independent lines in $\mathbb{R}^3$.*

"Then, the following are equivalent:"

a) *$\ell_0, \ldots, \ell_3$ are quadratically dependent*

b) *$\ell_0, \ldots, \ell_3$ are semiconcurrent*

c) *$\ell_0, \ldots, \ell_3$ satisfy either of the following:*

- *they are concurrent*
- *they meet in pairs and the planes these pairs generate meet in the line through the intersection points*
- *they belong to one ruling of a quadratic surface*

Proof. The proof requires the following fact, which is a simple exercise:

two lines $\{p + tn\}$ and $\{q + tm\}$ are non-skew if

$$(p \times n) \cdot m + (q \times m) \cdot n = 0 \tag{17}$$

Let $n_0, \ldots, n_3$ be directional vectors of $\ell_0, \ldots, \ell_3$ respectively, such that $\Sigma_0^3 n_i = 0$. Recall that by the definition of directional independence we have that any three of n_i's are linearly independent. And choose any point $p_i \in \ell_i$ for $i = 0, \ldots, 3$.

a $\Rightarrow$ b By hypothesis, we have that $\Sigma_0^3 p_i \times n_i = 0$. Let $\ell = \{q + tm\}$ be non-skew to ℓ_1, ℓ_2, ℓ_3. Using (17), we have that

$$(p_0 \times n_0) \cdot m + (q \times m) \cdot n_0 = \left(-\sum_{i=1}^{3} p_i \times n_i \right) \cdot m + (q \times m) \cdot \left(-\sum_{i=1}^{3} n_i \right)$$

$$= -\sum_{i=1}^{3} [(p_i \times n_i) \cdot m + (q \times m) \cdot n_i] = 0$$

Then, by (17), ℓ is also non-skew to ℓ_0. Since any other case is analogous; we may conclude that $\ell_0, \ldots, \ell_3$ are semiconcurrent.

b $\Rightarrow$ c If three of the lines meet, the fourth one must pass through the common point. Otherwise, semiconcurrence is easily contradicted. And the first possibility holds.

Suppose that ℓ_0 and ℓ_1 meet at a point q, say but $q \notin \ell_2, \ell_3$. Let Π be the plane through q and ℓ_2. Then, $\ell_3 \subset \Pi$ because any line from q to ℓ_2 also meets ℓ_3. Then, ℓ_2 meets ℓ_3 because they are not parallel. We also have that the plane ℓ_2 and ℓ_3 generate (namely, Π), contains $\ell_0 \cap \ell_1$. Reversing the roles of the pairs, we conclude with the second possibility.

Suppose finally, and generically, that $\ell_0, \ldots, \ell_3$ are pairwise disjoint.

It is classically known, see for instance the first chapter of [3], that three lines in general position, such as ℓ_1, ℓ_2, ℓ_3, are rules of a unique quadric surface. Thinking in the Projective Closure of $\mathbb{R}^3$ to make things precise, the construction reads as follows. For each $p \in \ell_1$, let Π_p be the plane through p and ℓ_2, and ℓ_p be the line through p and $\ell_3 \cap \Pi_p$. Observe that ℓ_p meets ℓ_1, ℓ_2 and ℓ_3, and that $\{\ell_p\}_{p \in \ell_1}$ is the set of all such lines, which could be parametrized by either of the three lines. Well, $S = U_{p \in \ell_1} \ell_p$ is a quadric surface expressed as the union of the lines in one of its rulings. By construction, $\ell_1, \ell_2, \ell_3 \subset S$, so that they must be rules in the other ruling . Semiconcurrence easily implies that $\ell_0 \subset S$ and thus that it belongs to the ruling of ℓ_1, ℓ_2, ℓ_3.

c $\Rightarrow$ a We must prove that $\Sigma_0^3 p_i \times n_i = 0$. Suppose that $\{q + tm\}$ is a line which is non-skew with $\ell_0, \ldots, \ell_3$. Then, using (17), we have

$$\left(\sum_{i=0}^{3} p_i \times n_i \right) \cdot m = \sum_{i=0}^{3} (m \times q) \cdot n_i = (m \times q) \cdot \left(\sum_{i=0}^{3} n_i \right) = 0$$

The proof concludes by observing that in either of the cases of (c), there are at least three linearly independent vectors m satisfying the above. Indeed, for each n_i (acting as m) one can easily find an appropriate q, and these suffice by our directional independence hypothesis.

With this proposition in mind, Theorem 1, stated in Section 1 as our main result, becomes a simple formal consequence of Theorems 2 and 3 of Section 2.

5. Immobilization of Convex Bodies

In this section we briefly analyse some necessary conditions for four points to immobilize a 3-dimensional convex body.

Let K be a convex body in $\mathbb{R}^3$, and let $p_0, \ldots, p_3$ be points in ∂K. Assume that $p_0, \ldots, p_3$, immobilize K.

Suppose that $\ell_0, \ldots, \ell_3$ are normal lines to K through $p_0, \ldots, p_3$ respectively; that is, the orthogonal plane to ℓ_i through p_i, Π_i say, is a K-supporting plane. By Lemma 1, the planes Π_i, define a tetrahedron T with $p_0, \ldots, p_3$ interior to its faces, and such that $K \subset T$. Since $\mathbb{C}T \subset \mathbb{C}K$, then $p_0, \ldots, p_3$ immobilize T. Therefore, Theorem 1 yields:

Corollary 1. *If $p_0, \ldots, p_3 \in \partial K$ immobilize the convex body K, then any set of normal lines to K at $p_0, \ldots, p_3$ is quadratically dependent.*

Now, let $\ell_0, \ldots, \ell_3$ be fixed normal lines to K at $p_0, \ldots, p_3$. Proposition 2 gives us three cases to analyze.

Case 1: (The generic case) If $\ell_0, \ldots, \ell_3$ are pairwise disjoint, then we may further imply that the points $p_0, \ldots, p_3$ are *regular*, that is, they have a unique normal line to K. Indeed, consider the quadric surface S of which $\ell_0, \ldots, \ell_3$ are rules. Since it is defined by any three of the lines, then any other line through p_0, say, other than ℓ_0, is not a rule of S. Therefore it is quadratically independent with ℓ_1, ℓ_2, ℓ_3, and, according to the corollary, it can not be normal to K.

With similar arguments to the ones used in the generic case, it is easy to see that if we want to immobilize using non-regular (or "corner" points), they must be chosen very carefully:

Case 2: If ℓ_0 and ℓ_1 meet, while ℓ_2 and ℓ_3 meet elsewhere, then in each pair one point may be non-regular. But in this case, lets say that p_0 is non-regular, we have that ℓ_0 and ℓ_1 intersect precisely at p_0, and all the normal lines to K at p_0 form a "linear interval" in the plane through ℓ_0 and ℓ_1. (Thus, the "corner" is orthogonal to the plane.)

Case 3: If the four normals are concurrent then at most one point is non-regular; and if this is the case, then that point is the meeting point.

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