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David Mayer Foulkes

IMMOBILIZATION OF N-DIMENSIONAL GEOMETRICAL FIGURES

Abstract. We treat for n-dimensional bodies the immobilization problem introduced by Kuperberg and Papadimitriou, giving the zeroeth, first and second order conditions for fixing, as well as a geometric characterization of the first order condition. We show the equivalence of the geometrical and mechanical conditions of fixing. We show that generically, C^1 bodies may be fixed by $n+1$ points. Also, there is a $C^{1,1}$ neighbourhood of S^n in which a body not admitting threads (sets of trapping points which may slide along the surface) may be fixed by $n+1$ points. We show that a star-shaped body trapped by a set P either may be fixed by a set similar to P, or admits a thread generated by P.

Introduction

Immobilization problems where introduced by W. Kuperberg [K] and Papadimitriou [MNP1]. They were motivated by grasping problems in robotics [MNP1, 2]. Interest then developed in the purely geometrical aspect of the problem. Focusing on smooth convex curves, in [BMU], geometrical conditions were obtained for the first and second order conditions of immobilization for plane figures, and it was proved that analytic convex figures other than the disk may be fixed by three points. In [BFMM], focusing instead on tetrahedra, the first order necessary condition was shown to imply the second order sufficient condition in the three dimensional case. In [M], the theorem on Mondriga matrices necessary for this result was generalized to n dimensions. Also, the Kuperberg conjecture was proved in the two dimensional case: every C^2 strictly convex figure may be fixed by three points satisfying the second order condition unless it is the disk.

Here, we are interested in the problem from the n-dimensional perspective. We give a geometrical interpretation of the first order condition, and give the second order condition in the C^2 case. We show the equivalence of the geometrical and mechanical conditions of fixing. We show a result relating trapping to fixing, and also show that $C^{1,1}$ bodies in the neighbourhood of the sphere may be fixed by $n+1$ points unless they admit a thread (analogous to the thread of a screw).

2. Definitions

Let us make precise the concepts of "immobilization" and "trapping". We follow the notation found in [BFMM]. Let $\mathscr{E}$ be the Lie Group of orientation preserving isometrics of Euclidean space $\mathbb{R}^n$. Given any two sets $X, Y \in \mathbb{R}^n$ define the motions of X in Y to be

$$\mathscr{E}(X, Y) = \{g \in \mathscr{E} \mid g(X) \subset Y\}.$$

Throughout this article, let $K \subset \mathbb{R}^n$ be a compact body with non-empty interior. Denote by Int$K$ the interior of K, and by $\mathbb{O}K$ its "outside", that is, $\mathbb{O} = \mathbb{R}^n \setminus \mathrm{Int}K$, so that $K \cap \mathbb{O}K = \partial K$.

2.1. Definition. We say that P *immobilizes* (or *fixes*) K if $P \subset \mathbb{O}K$ and the identity map id $\in \mathscr{E}$ is an isolated point component of $\mathscr{E}(P, \mathbb{O}K)$ (with respect to path connectedness). We say that P *traps* K if $P \subset \mathbb{O}K$ and the connected component of id $\in \mathscr{E}$ is compact.■

The exceptional cases to immobilization (such as those posed by spheres or screws) are cases in which points which almost fix a body can slide along its surface. In these cases we say K admits a thread.

2.2. Definition. We say that K admits a *global (local) thread* (on its surface) if there exists a set $P \subset \mathbb{O}K$ which traps K and which satisfies the property: for every $g \in \mathscr{E}(P, \mathbb{O}K)$ in the connected component of id $\in \mathscr{E}$ (or only in a neighbourhood of id), $g(P) \cap \partial K \neq \phi$. We say that P generates a thread.■

It is clear that each $g(P)$ traps K. The idea is that the union of sets $g(P) \cap \partial K$ is what in simple cases such as the surface of a screw we call a thread.

3. The Zeroeth, First and Second Order Conditions

We are interested in the conditions under which a set of $n+1$ points $P = \{p_0, \ldots, p_n\}$ fixes an n-dimensional body K at differentiable points on the boundary. Let the set of outward normals corresponding to these points be $N = \{N_0, \ldots, N_n\}$. The simplest necessary condition for fixing, which we refer to as the zeroeth order condition, is that the points under consideration fix the body when motions are restricted to translations.

3.1. Proposition. A necessary and sufficient condition for the points P to fix a C^1 body K up to translations (the *zeroeth order* condition) is that any proper subset of N is linearly independent and that there exist positive constants $a_0, \ldots, a_n > 0$ for which the set N of normals satisfies $\Sigma_0^n a_i N_i = 0$.

Proof. Let us denote translations by vectors b. P fixes K up to translations if for every b some point p_i penetrates the interior of K when translated in the direction b:

$$\forall\, b \neq 0\ \exists\ \leq i \leq n \mid N_i \cdot b < 0. \tag{3.1.1}$$

This implies that any proper subset of N is linearly independent. Otherwise there exists (renumbering) some set $\{N_0, \ldots, N_{n-1}\}$ contained in a hyperplane, so for some vector b, $b \cdot N_0 = \ldots = b \cdot N_{n-1} = 0$. Then (replacing $-b$ by b if necessary) $N_n \cdot b \geq 0$, contradicting 3.1.1. Also, 0 must be in the interior of the convex hull of N. Otherwise there exists a hyperplane separating N from 0, that is, a vector b such that $N_i \cdot b \geq 0$.

Conversely, if for any $b \neq 0$ 3.1.1 is false, then $N_1 \cdot b \geq 0$. But the independence condition on N implies these quantities cannot all be zero. Thus $0 = \Sigma_0^n a_i N_i \cdot b > 0$, a contradiction.■

Whenever we suppose a set of points P satisfies the zeroeth order condition we shall write $n_i = a_i N_i$; $\Sigma_0^n n_i = 0$. The $a_i > 0$ are defined up to a constant, but for definiteness we choose

$$a_i = (-1)^i \det\left(N_0 \ldots \hat{N}_i \ldots N_n\right) \tag{3.1.2}$$

(the hat means "omit"), with the numbering chosen so $N_1, \ldots, N_n$ has the canonical orientation.

To develop the first and second order conditions for fixing we consider the following geometrical construction. Suppose a set of points P fixes a body K up to translations, and that in a neighbourhood of the points P ∂K is twice differentiable, so that the second fundamental form exists. It turns out that for any rotation, there exist corresponding translations and homothetic scale changes which cause each point of P to *slide* along ∂K. Then, if for all rotations the necessary scale change is an increase, P must fix K.

We represent the second fundamental form of the surface ∂K with normal N by $B(x) = D_x N$ and also write $B(x, y) = y^T D_x N$.

3.2. Proposition. Suppose $P = \{p_0^0, \ldots, p_n^0\}$ fixes a C^2 figure K up to translations, and $\Sigma_0^n n_i^T p_i \neq 0$ on ∂K (this is true for starshaped K). For any C^2 path of orthogonal transformations $R(t)$ with $R(0) = I$ define the vectors $p_i(t)$, $b(t)$ and the scale factor $\sigma(t)$ by

$$p_i = \sigma R\,(p_i^0 + b),\ n_i^T p'_i = 0,\ i = 0, \ldots, n, \qquad b(0) = 0,\ \sigma(0) = 1. \tag{3.2.1}$$

These are equivalent to the system of o.d.e.'s

$$p'_i = \sigma' \sigma^{-1} p_i + A p_i + \sigma R b' \tag{3.2.2}$$

where $A = R' R^{-1}$

$$\sigma' = -\sigma \frac{\Sigma_0^n n_i^T A p_i}{\Sigma_0^n n_i^T p_i}, \tag{3.2.3}$$

and b' is obtained by solving

$$n_i^T \sigma(Rb') = -(\sigma' \sigma^{-1} n_i^T p_i + n_i^T A p_i),\ i = 0, \dots, n\ . \tag{3.2.4}$$

The system of o.d.e's obtained after substituting σ' and b' in 3.2.2 (using 3.1.2 for the definition of a_i) has a unique solution in a neighbourhood of $t = 0$, in which $p_0(t), \dots, p_n(t)$ satisfy the zeroeth order condition.

P fixes K if for every path $R(t)\ \sigma$ increases arbitrarily close to $t = 0$, for $t > 0$ and $t < 0$, meaning

$$\forall\, \varepsilon > 0\ \exists\ t_i > 0,\ t_2 < 0\ \mid\ |t_i| < \varepsilon,\ \sigma(t_i) > 1,\ i = 1, 2.$$

The *necessary first order condition* for this to be the case is

$$\Sigma_0^n\, p_i \wedge n_i = 0 \tag{3.2.5}$$

Given this condition, the *sufficient second order condition* is

$$\Sigma_0^n\, a_i \Big((AN_i) \cdot (Ap_i) - B_i\, (p_i', p_i') \Big) > 0\ , \tag{3.2.6}$$

where B_i is the second fundamental form at p_i^0, $i = 0, \dots, n$. In terms of equations 3.2.2, 3.2.3 and 3.2.4, we may define

$$Q\,(A, A) = \Sigma_0^n\, a_i \Big((AN_i) \cdot (Ap_i) - B_i\, (p_i', p_i') \Big). \tag{3.2.7}$$

Q can be extended to a symmetric bilinear quadratic form.

Proof. Conditions 3.2.1 define the images of p_i^0 under a path of isometries preceded by a translation and followed by the application of a scale factor both defined uniquely under the condition that the points remain on the surface. The uniqueness is clear from the differential system, which is obtained as follows. Differentiating in 3.2.1

$$p_i' = \sigma' R\,(p_i^0 + b) + \sigma R'\,(p_i^0 + b) + \sigma R b'.$$

Substituting $p_i^0 = \sigma^{-1} R^{-1} p_i - b$ we obtain 3.2.2. Therefore

$$0 = n_i^T p_i' = \sigma' \sigma^{-1} n_i^T p_i + n_i^T A p_i + \sigma n_i^T R b'$$

so

$$0 = \sigma' \sigma^{-1}\, \Sigma_0^n\, n_i^T p_i + \Sigma_0^n\, n_i^T A p_i\ ,$$

implying 3.2.3 and 3.2.4. While a_i and $\Sigma_0^n n_i^T p_i$ remain different from zero the full o.d.e. system for p_i, σ, b, a_i, a may be obtained as rational expressions with non-zero denominators [to obtain b the inverse matrix of $(N_1 \dots N_n)$ is involved] ensuring the existence and uniqueness of solutions in a neighbourhood of $t = 0$.

It is not hard to see for any path of rotations $R(t)$ that a path of isometries shifting the points p_i^0 without entering the interior of K exists if and only if the scale factor keeping them on the surface does not increase on both sides of $t = 0$, arbitrarily closely.

The necessary first order condition is $\sigma'(0) = 0$ for every path $R(t)$, which is equivalent to the condition that $\Sigma_0^n n_i^T A p_i = 0$ for every antisymmetric matrix A, and therefore to 3.2.5. Given the first order condition, the second order sufficient condition is

$$(\ln\sigma)''\,|_0 = -\frac{\Sigma_0^n n_i^T A p_i' + n_i'^T A p_i}{\Sigma_0^n p_i \cdot n_i}\,|_0 - \frac{\Sigma_0^n n_i^T A' p_i}{(\Sigma_0^n p_i \cdot n_i)^2}\,|_0 > 0\,.$$

We examine term by term. The first term gives

$$\Sigma_0^n n_i^T A p_i' = \Sigma_0^n n_i^T A A p_i = -\Sigma_0^n (A n_i) \cdot (A p_i)$$

The second term gives

$$\Sigma_0^n n_i'^T A p_i = \Sigma_0^n (a_i' N_i + a_i N_i')^T p_i' = \Sigma_0^n a_i B_i (p_i', p_i').$$

The last term is zero because $\Sigma_0^n n_i p_i^T$ is symmetric and

$$(A')^S = (R' R^{-1})'^S = \frac{1}{2}(R' R^T + R^T R')' = \frac{1}{2}(RR^T)'' = \frac{1}{2} I'' = 0\,.$$

Hence the second order sufficient condition is equivalent to 3.2.6. Q may be extended to a symmetric bilinear quadratic form because 3.2.5 is equivalent to the symmetry of the matrix $\Sigma_0^n n_i p_i^T$.

For $n = 2$ it is enough to consider the smooth path of rotations

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \text{for wich } A = R' R^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

so $A' = 0$. The sign of the derivative $(\ln\sigma)''$ coincides with the sign of the expression in 3.2.6, in a neighbourhood of $t = 0$.■

3.3. Definition. We say that a body K held by a set of $n + 1$ points P is fixed *firmly* if it satisfies the zeroeth, first, and second order conditions.■

The first order condition 3.2.5 has the following geometric characterization. We write $< A>$ for the vector subspace generated by the set or list of vectors A.

3.4. Theorem. Let p_i, n_i, $0 \leq i \leq n$ be a set of $n + 1$ points and directions defining lines L_i, and suppose that the normals satisfy the zeroeth order condition (see 3.1). Then $\Sigma_0^n p_i \wedge n_i = 0$ if and only if each $n - 2$ dimensional plane which intersects or is parallel to n of the lines L_i intersects or is parallel to the remaining line.

Proof. Suppose $\Sigma_0^n p_i \wedge n_i = 0$. Let Q^{n-2} be a $n - 2$ dimensional plane containing linearly independent directions $v_1, \ldots, v_{n-2}$ and a point $q \in Q^{n-2}$. Suppose that L_i intersects Q^{n-2}, $i \neq j$. Then $q - p_i \in <n_i, v_1, \ldots, v_{n-2}>$ so that

$$(q - p_i) \wedge n_i \wedge v_1 \wedge \ldots \wedge v_{n-2} = 0.$$

The same equation is obtained if L_i is parallel to Q^{n-2} since in this case n_i is a linear combination of $v_1, \ldots, v_{n-2}$. Hence

$$(q - p_j) \wedge n_j \wedge v_1 \wedge \ldots \wedge v_{n-2} = -\sum_{i \neq j}^{n} (q - p_i) \wedge n_i \wedge v_i \wedge \ldots \wedge v_{n-2} = 0,$$

since $\Sigma_0^n n_i = 0$ and $\Sigma_0^n p_i \wedge n_i = 0$. We infer that L_j also intersects or is parallel to Q^{n-2}, according to $n_j \wedge v_1 \wedge \ldots \wedge v_{n-2}$ being different or equal to zero.

We prove the converse for $n = 2$ and then reduce the general case to this one. For $n = 2$ there exists some point q at which L_0 and L_1 intersect, since they are not parallel. Therefore L_2 also goes through q so we have $p_i = q + \alpha_i n_i$ for $i = 0, 1, 2$, implying

$$\Sigma_0^2 p_i \wedge n_i = \Sigma_0^2 (1 + \alpha_i n_i) \wedge n_i = 0 .$$

For $n \geq 3$ express $\omega = \Sigma_0^n p_i \wedge n_i$ in terms of the basis $n_1, \ldots, n_n$. If we had $\omega \neq 0$, renumbering if necessary, $n_1 \wedge n_2$ has a non-zero coefficient so $\pi(\omega) \neq 0$, where $\pi : E^n \to E^2$ is the projection along $<n_3, \ldots, n_n>$. By the linear independence conditions on the normals, $\pi(L_0)$, $\pi(L_1)$, $\pi(L_2)$ satisfy the intersection hypothesis for $n = 2$, so we can conclude that $0 = \Sigma_0^2 \pi(p_i) \wedge \pi(n_i) = \pi(\omega)$, which is a contradiction.■

In a communication to the author, Professor Elmer Reese points out that an alternative interpretation of the first order condition is that the projective coordinates of the lines L_i are linearly dependent (via the Plucker embedding).

We say that the lines L_i corresponding to points satisfying the first order condition are *concurrent* for $n = 2$ and *semiconcurrent* for $n \geq 3$.

An interesting result is that the geometrical conditions for fixing coincide with the mechanical conditions.

3.5. Theorem. Suppose for a body K in $\mathbb{R}^3$ a set of 4 points P satisfies the zeroeth and first order conditions. If forces F_i are applied inwardly along the normals at the points P, so as to add up to zero, these must be proportional to n_i, and the resultant torque is zero. Suppose further that the mechanical system delivering forces F_i, is subject to motions of P and K, with movements of P restricted to being similar, in such a manner that F_i continue to be applied inwardly along the normals N_i. Then if the second order condition is satisfied, any motion of K must do work against F_i.

Proof. If the forces applied are $F_i = \phi_i N_i$ and $\Sigma_0^3 F_i = 0$, ϕ_i must be a multiple of a_i, since N_i satisfy the zeroeth order condition. The resultant torque is therefore a multiple of $\Sigma_0^3 p_i \wedge n_i = 0$. The geometry of the second order condition implies that any motion of K will necessitate a positive change of scale of P, which will do work against F_i.■

4. Some General Theorems for the n-dimensional Case

We first relate the concepts of fixing and trapping. If a star-shaped body is trapped by a closed set P, by diminishing the scale of P and shifting it isometrically we must eventually fix K, unless special features exist on the surface of K, which we have called threads.

Let us write $D_x(r) : \mathbb{R}^n \to \mathbb{R}^n$ for dilations with a scale r centered at x: $D_x(r)(y) = r(y - x) + x$. Recall that any $\theta \in \mathscr{E}$ may be written as $\theta x = Rx + b$ where R is orthogonal, and b represents a translation.

4.1. Theorem. Suppose K is star-shaped about some point in its interior, for definiteness the origin, so $D_0(r)K \subset K$. Suppose a closed set $P \subset \mathbb{R}^n$ traps K. Then there exists an isometry $\phi \in \mathscr{E}$ and a scaling factor $r \in (0, 1]$ such that either the reduced image $P' = \phi(D_0(r)P)$ of P immobilizes K or it generates a thread on the surface of K.

Proof. Let $r = \inf\{s \in [0, 1] \mid \exists\, \phi \in \mathscr{E} : \phi(D_0(0)P) \text{ traps } K\}$ be the infimum of set of scaling factors for which some reduced image of P lies outside K and traps K. Then there exist some sequences $s_i \in [r, 1]$ tending to r, and $\phi_i \in \mathscr{E}$, where $i \in \mathbb{N}$, for which $\phi_i(D_0(s_i)P) \subset \mathbb{C}K$ and traps K. ϕ_i clearly belong to a compact subset of $\mathscr{E}$ and so there is a convergent subsequence tending to some $\phi \in \mathscr{E}$ for which $P' = \phi(D_0(r)P) \subset \mathbb{C}K$. Since $0 \in \text{int}K$, $r > 0$. Hence P' is similar to P. We show P' traps K. Suppose instead that there exists a path $\theta : [0, \infty) \to \mathscr{E}(P', \mathbb{C}K)$ with $\theta(0) = \text{id}$ for which $\theta(t) \to \infty$ (that is, θ has a large translation component). Since K is star-shaped about 0, $\mathbb{C}K \subset \mathbb{C}(D_0(r)K)$, so

$$\theta(t) \cdot \phi(D_0(r)P) \subset \mathbb{C}K \subset \mathbb{C}(D_0(r)K).$$

There exist $R(t)$, $b(t)$ for which $\theta(t) \cdot \phi x = R(t)x + b(t)$. Then

$$R(t)rP + b(t) \subset rK$$

which implies

$$R(t)P + r^{-1} b\,(t) \subset K\,.$$

Letting $\chi\,(t)x = R(t)x + r^{-1}b\,(t)$, we have a path $\chi : [0, \infty) \rightarrow \mathscr{C}(P, \mathbb{C}K)$ showing that P does not trap K.

Suppose now that id is not an isolated point component of $\mathscr{C}(Q, \mathbb{C}K$. If for any $g \in \mathscr{C}$ in this component $g\,(P) \cap \partial K = \phi$, then a smaller scaling factor than r would exist, since P is closed. hence P' generates a thread.■

The next theorem proves that one method of fixing bodies, which works generically for C^1 bodies K, is to find the largest ball inscribed in ∂K. This is the gateway to the general theorem of fixing convex bodies in dimension 2, proved in [M].

4.2. Theorem. *a)* Suppose a closed ball contained in a C^1 body $K \subset \mathbb{R}^n$ touches it only on an open semisphere. Then there is a bigger ball contained in K.

b) Suppose an inscribed closed ball in a C^1 body K contains on its intersection with ∂K a set of $n + 1$ points $P = \{p_0^0, \ldots, p_n^0\}$ whose corresponding normals N_i satisfy the zeroeth order condition. Then either P fixes K or it generates a thread.

c) C^1 compact bodies K with non-empty interiors whose largest inscribed balls B have intersections with ∂K containing $n + 1$ points P fixing K are C^1 dense.

Proof. *a)* By hypothesis there exists a direction h such that $B \cap \partial K \subseteq \{x : x \cdot h < 0\}$. Define on the upper hemisphere $\{x \in \partial B : x \cdot h \geq 0\}$ the continuous function $\rho(x) = sup\,\{r : rx \in K\}$. ρ must attain its minimum $\rho_0 > 0$. Let C be the convex hull of

$$\{x \in B \mid x \cdot h < 0\} \cup \{\rho_0\, x \mid x \in B\} \subseteq K.$$

Since K is convex, $C \subseteq K$. It is also clear that C contains a ball slightly bigger than B.

b) We shall show that the P fixes the sphere itself except for rotations; hence K is trapped a fortiori. Since the sphere is invariant under rotations, the only relevant paths of isometries with one endpoint at the identity are translations. But in any translation direction h, since the normals satisfy the zeroeth order condition, at least some $h \cdot N_i > 0$, implying $p_i\,(t)$ penetrates the sphere. It follows that if P does not generate a thread on K, it fixes K, because if a path of rotations $R\,(t)$ defines a scaling path $\sigma\,(t)$ which cannot decrease (since then some path p_i enters the ball, which is a subset of K) or remain constant (this would define a thread) it must increase arbitrarily closely to $t = 0$ on either side.

c) Any compact body K with non-empty interior has a largest inscribed ball B. The intersection $I = \partial B \cap \partial K$ may not be contained in an open semisphere by (a). If it is contained in a closed semisphere, $n + 1$ points on B, each arbitrarily close to points in I, may be selected so that they are not all contained in a semisphere. Then K may be modified to a C^1 body arbitrarily close to the original for which B is the largest inscribed ball and for which I does not generate a thread. The $n+1$ points satisfy the zeroeth order condition by construction and so by (b) fix K.■

The next theorem shows a general condition under which, given a particular simplex S, there exists a set of $n+1$ points P forming a simplex similar to S (with the same orientation) which fix K, unless K admits a thread generated by P.

4.3. Theorem. Let K be a C^1 body and $S = \{s_0, \ldots, s_n\} \subset \mathbb{R}^n$ be a set of points defining a simplex. Suppose that for every set of points $P = \{p_0, \ldots, p_n\} \subset \partial K$ defining an inscribed simplex similar to S and with the same orientation, the corresponding set of normals N satisfies the zeroeth order condition. Then at least one of these sets P fixes K, unless K admits a thread generated by P.

Proof. For each rotation, there exists a largest scaling factor σ for which for some orientation preserving isometry ϕ, $\sigma\phi(S) \subset K$. Therefore the set of inscribed simplexes similar to S and with the same orientation is non-empty. Take the infimum

$$\sigma = \inf\{\sigma > 0 \mid \exists\, \phi \in \mathscr{C} : \sigma\phi(S) \subset \partial K\}.$$

Again taking a convergent subsequence we have some corresponding $\phi \in \mathscr{C}$ for which $P = \sigma\phi(S) \subset \partial K$. Now for every sliding of P (see §3.2) generated by a path of rotations $R(t)$, the scaling factor $\sigma(t)$ has a global minimum at $t = 0$. If for some path σ is constant on some interval containing 0, K admits a thread generated by P. Otherwise for every path $R(t)$ σ increases arbitrarily close to $t = 0$ so P fixes K.■

4.4. Theorem. Let $S = \{s_0, \ldots, s_n\} \subset \mathbb{R}^n$ be a set of points defining a simplex for which, for every set of points $P = \{p_0, \ldots, p_n\} \subset S^n$ defining a simplex similar to S and with the same orientation, inscribed on the sphere, the corresponding set of normals N satisfies the zeroeth order condition. There is a $C^{1,1}$ neighbourhood of bodies close to the sphere S^n for which S has the same property.

Proof. Since the sphere is convex, there is a $C^{1,1}$ neighbourhood of bodies close to it which are convex and which form a simply connected domain in $\mathbb{R}^{n+1}$. Thus we restrict our attention to such bodies, for which there exists a $C^{1,1}$ concave function ϕ on $\mathbb{R}^{n+1}$ for which $\partial K = \phi^{-1}(1)$, $\phi(0) = 0$, $\nabla\phi(0) = 0$. One way of finding such a function is to consider the first eigenfunction ϑ of the Laplacian with Dirichlet boundary conditions, which is convex [C, Chp I, §5 remark 3] and has a unique maximum at some interior point, which we rescale to 1. Fixing the origin at the maximum we can take $\phi = 1 - \vartheta$, and extend it to a $C^{1,1}$ concave function on $\mathbb{R}^{n+1}$ satisfying the desired conditions. Define

$$\psi(t) = t\phi + (1-t)|x|^2 \qquad t \in [0, 1]$$

Then $\psi(t)(0) = 0$, $\nabla\psi(t)(0) = 0$. We deform K to the sphere by letting $\partial K(t) = \psi(t)^{-1}(1)$. Suppose now that we are given some points $P = \{p_0^1, \ldots, p_n^1\}$ forming a simplex inscribed in ∂K. We wish to find a path of inscribed homothetic simplexes given by $p_i(t)$ with $p_i(1) = p_i^1$, $p_i(0) \in S^n$. Thus we require

$$p_i(t) = \sigma(t)(p_i^1 + b(t))$$

which implies

$$p_i' = \sigma' \sigma^{-1}(p_i + b')$$

We have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\psi(t, p_i(t)) = \psi_t + \nabla\psi \cdot p'_i,$$

so

$$-\psi_t |\nabla\psi|^{-1} = N_i \cdot p_i' = \sigma' \sigma^{-1}(N_i \cdot p_i + N_i \cdot b')$$

Hence

$$\sigma' = -\sigma \sum_{i=0}^{n} a_i (\psi_t |\nabla\psi|^{-1})(p_i) \left(\sum_{i=0}^{n} n_i \cdot p_i \right)^{-1}$$

and $b' = F((\psi_t |\nabla\psi|^{-1})(p_i), p_i, N_i, \sigma)$ (which includes matrix inverses of matrices of vectors N_i). Thus the differential equation is well defined, and has local solutions since $\nabla\psi$ is Lipchitz because ψ is $C^{1,1}$. Observe that

$$|\nabla\psi|^2 = |t\nabla\phi + 2(1-t)x|^2 = t^2|\nabla\phi|^2 + 4t(l-t)<\nabla\phi, x> + 4(1-t)^2|x|^2$$

is positive on $\partial K(t)$ since each of its terms is positive at every point except for the origin, which is never on $\partial K(t)$.

If we solve with $\phi = |x|^2$, the solution exists on the interval [0, 1], given by $\sigma(t) = 1$, $b(t) = 0$. This solution has the property $a_i =$ constants > 0. By the continuity of solutions of uniformly Lipchitz differential equations, and the compactness of the set of inscribed simplexes P, there is a $C^{1,1}$ neighbourhood of functions ϕ close to $|x|^2$ for which solutions satisfying $a_i > 0$ also exist on the whole interval, for given P.

But these functions ϕ define a $C^{1,1}$ neighbourhood of bodies of the sphere, each having the desired property.■

4.5. Corollary. For any simplex S with the property that any oriented similar copy inscribed on the sphere has normals satisfying the zeroeth order condition (this holds for a neighbourhood of equilateral simplex), there is a $C^{1,1}$ neighbourhood of bodies K close to the sphere S^n, any of whose elements is either fixed by an oriented similar copy P of S, or admits a thread generated by P.■

It is clear that in the general fixing problem, the boundary for slidings formed by the zeroeth order condition plays an important role.

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