CIDE

NÚMERO 50

David Mayer-Foulkes

THE CORRELATION DIMENSION RATIO: COMPARISON WITH THE DBS STATISTIC

The Correlation Dimension Ratio: comparison with the DBS statistic

David Mayer-Foulkes[1]
División de Economía, CIDE

Abstract

In "An Alternative Correlation Statistic", the author compares the Correlation Dimension Ratio (CDR) with Grassberger and Procaccia's Correlation Dimension, (CD) showing it to be equally useful for distinguishing between deterministic and stochastic processes, while at the same time providing a test for the IID null. The CDR also eliminates the CD's bias towards low dimension in relatively short time series (this bias accounts for spurious low-dimensional results of 5 to 7 in the literature). Here, the CDR, and a related statistic, the Correlation Ratio Logarithm (CRL), are compared to the DBS statistic in a Monte Carlo experiment measuring the moments of their distributions. The CDR, followed by the CRL, are found to be consistently closer to the normal distribution, more sensitive to stochastic structure, and less heteroskedastic as ε varies.

1 Introduction

In recent years the application of the Grassberger and Procaccia (G&P) correlation dimension (CD) (1983) in economics has been of interest. The CD measures fractal dimension and can distinguish between stochastic and

[1]Email: mayerfou@dis1.cide.mx

deterministic (e.g. chaotic) processes. Brock, Dechert and Scheinkman (1987) then developed a related statistic (referred to as the DBS statistic) testing for the IID null whose applications include testing for nonlinearity in stochastic processes. In a related article (Mayer, 1995), we define a similar statistic which we refer to as the Correlation Dimension Ratio (CDR). It has similar applications to the DBS statistic, and in addition serves as a good numerical method for the calculation of the CD for series having correlation dimensions greater than one. In particular, it removes the bias towards low dimension present in applications to time series of lengths around 500 (about the minimim length for which meaningful results can be obtained) and shows that some of the ambigous or low-dimensional results reported in the literature (typically dimensions between 5 and 7) are spuriously induced by the CD.

In that paper, several tests compare the CDR to the CD, showing it to provide a more powerful test for presence of non-linear stochastic structure. The purpose of this paper is to compare the CDR to the DBS statistic. By carrying out a Monte Carlo experiment measuring the moments of the distributions, we find that the CDR is closer to the normal distribution than the DBS statistic, more sensitive to stochastic structure, and that its variance varies less with ε, the measure of distance involved in the correlation dimension.

2 The Correlation Dimension Ratio (CDR)

Let $\mathcal{A} = \{a_t : t = 1,\ldots,N\}$ be a finite series of real numbers. For each m = 1,2,...,N, $\tau \in \mathbb{N}$, write

$$H(\mathcal{A},m,\tau) = \{(a_t,\ldots,a_{t+(m-1)\tau}) : t = 1,\ldots,N-(m-1)\tau\} \subseteq \mathbb{R}^m \quad (2.1)$$

for the set of τ-lagged m-histories of $\mathcal{A}$. Abbreviating $H = H(\mathcal{A},m,\tau)$, and letting #H be the cardinality of H, let $C_m(\varepsilon) : (0,\infty) \to [0,1]$, given by

$$C_m(\varepsilon) = \frac{\#\{(x,y) \in H\times H : x \neq y, |x-y| < \varepsilon\}}{\#H(\#H-1)} \quad (2.2)$$

be the relative frequency with which the distance between pairs of vectors in H is less than ε. Here $\| \ \|$ is the maximum norm of $\mathbb{R}^m$.

Definition. The *correlation dimension ratio* (CDR) of such a sequence $\mathcal{A}$ with lags of order τ is (in terms of the notation above)

$$\mathrm{CDR}_\tau = \lim_{m\to\infty}\mathrm{CDR}(m); \quad \mathrm{CDR}(m) = \lim_{\varepsilon\to 0}\mathrm{CDR}(m,\varepsilon,\tau),$$
$$\mathrm{CDR}(m,\varepsilon,\tau) = \frac{\log C_m(\varepsilon)}{\log C_1(\varepsilon)}. \tag{2.3}$$

Grassberger and Procaccia's *correlation dimension* (CD) instead has log(ε) as numerator:

$$\mathrm{CD}_\tau = \lim_{m\to\infty}\mathrm{CD}(m); \quad \mathrm{CD}(m) = \lim_{\varepsilon\to 0}\mathrm{CD}(m,\varepsilon,\tau),$$
$$\mathrm{CD}(m,\varepsilon,\tau) = \frac{\log C_m(\varepsilon)}{\log\varepsilon}. \tag{2.4}$$

The DBS statistic is

$$\mathrm{DBS}(m,\varepsilon,\tau) = C_m(\varepsilon) - C_1(\varepsilon)^m. \tag{2.5}$$

We also introduce the *correlation ratio logarithm*,

$$\mathrm{CRL}(m,\varepsilon,\tau) = \log\left(\frac{C_m(\varepsilon)}{C_1(\varepsilon)}\right) - m. \blacksquare \tag{2.6}$$

We mention a few general points about these statistics.

Let $\{Z_j\}$ be a strictly stationary process and let $\mathcal{A} = \{Z_1,\ldots,Z_N\}$. Then we may regard CDR(m,ε,τ), CD(m,ε,τ), DBS(m,ε,τ), CRL(m,ε,τ) as random variables. By applying the technique of U-statistics (see Serfling, 1980) Brock and Baek (1991) show that DBS(m,ε,τ) is generically asymptotically normally distributed as $N \to \infty$, with mean 0. Their results extend straightforwardly to CDR(m,ε,τ), CD(m,ε,τ) and CRL(m,ε,τ) (see Mayer, 1995).

In this paper we shall use τ = 1, and remove τ from the notation. Observe that using τ > 1 is equivalent to replacing $\{Z_1,\ldots,Z_N\}$ with $\{Z_1,Z_{1+\tau},\ldots,Z_{1+(N-1)\tau}\}$, so that in our tests this would be equivalent to replacing the process Z with the iterated process.

By definition, CD(m) = CDR(m)CD(1). For stochastic processes, and for deterministic processes with CD ≥ 1, CD(1) = 1. Thus for these processes CD(m) = CDR(m). For deterministic processes with CD < 1, CD(m) = CD(1) < 1 so CDR(m) = 1. Hence CDR(m) distinguishes stochastic from deterministic processes just as CD(m), but will yield CDR(m) = 1 for cases in which CD ≤ 1. The theorems which establish these and other general properties of CD(m) also establish those of CDR(m) (see Brock and Dechert, 1987).

In practice, the CD of a time series is determined as follows. For each m, $CD(m) = \lim_{\varepsilon \to 0} CD(m,\varepsilon)$ is calculated as the coefficient in a linear regression of $\log C_m(\varepsilon)$ with respect to $\log\varepsilon$, in a region of small values of ε where the relation between both variables is approximately linear. $CDR(m) = \lim_{\varepsilon \to 0} CDR(m,\varepsilon)$ is similarly determined in a linear regression of $\log C_m(\varepsilon)$ with respect to $\log C_1(\varepsilon)$.

3 Comparing the Correlation Dimension Ratio with the DBS statistic

The purpose of this paper is to compare the CDR with the DBS statistic. Two important points arise when deciding how to compare them. The first is that the CDR uses data obtained for a region of values of ε, while the DBS statistic is defined for a fixed value of ε. There is no clear, practical way to combine the information obtained from several values of ε, because the variance depends on ε and its calculation is very lengthy (of the order of N CD calculations). The second is that, since the CDR is obtained by a regression rather than a division, its behavior at a fixed value of ε could be similar to $CRL(m,\varepsilon)$ as well as $CDR(m,\varepsilon)$.

The performance of the DBS statistic in particular depends on how well it approximates a normal distribution, because significance intervals are obtained by calculating the asymptotic variance. Normality, which is important per se, also has a bearing on the validity of the linear regression used to calculate the CDR. Thus our comparison of the statistics will be based on testing their distributions for normality. The CDR and the DBS statistic are both based on the same component random variables $C_m(\varepsilon)$. For fixed values of ε, they are obtained by different functional transformations which alter their probability distributions. Thus measuring the skewness and kurtosis of the distributions should provide a good test of the differences in normality.

Thus our method is to carry out a Monte Carlo experiment measuring the moments of the statistics CDR(m,ε), CRL(m,ε) and DBS(m,ε).

4 The Monte Carlo experiment

We carried out the experiment for six processes $Z^k = (Z^k_j: 1 \le j \le 500)$. In the context of dimension calculations, series with N = 500 terms are short. We compare for each Z^k how close the distributions of the dimension measures CDR(Z^k)(m,ε), CRL(Z^k)(m,ε), DBS(Z^k)(m,ε) are to the normal distribution, which they approximate asymptotically when $N \to \infty$.

The six processes Z^k are the following. Let u_j be IID normal random variables N(0,1) and v_j be IID uniformly distributed random variables. Define Z^1 to Z^6 by

$$
\begin{gathered}
Z^1_j = u_j; \qquad Z^2_j = v_j; \\
Z^3_j = |u_{j-1}|u_j \qquad Z^4_{j+1} = .6Z^4_j + |u_{j-1}|u_j; \\
Z^5_j = a_j u_j; \qquad Z^6_{j+1} = .6Z^6_j + a_j u_j;
\end{gathered} \tag{3.1}
$$

$1 \le j \le 500$, with initial values 0 where necessary and where a_j is a random variable with value 1 or 2 following a Markov process defined by the probabilities

$$P(a_{i+1} = i \mid a_i = j) = A_{ij}, \quad i,j \in \{1,2\}, \quad \text{where} \quad A = \begin{pmatrix} .2 & .8 \\ .8 & .2 \end{pmatrix}. \tag{3.2}$$

Our numerical procedure for the calculation of $C_m(\varepsilon)$ has been reported (Mayer, 1995). We obtain $C_m(\varepsilon_i)$ recursively in m using mainly short integer arithmetic, using an algorithm with low memory requirements. We use 256 values of ε, namely

$$\varepsilon_i = \frac{i}{256}\varepsilon^*, \quad i = 1,\ldots,256, \quad \text{where} \quad \varepsilon^* = \max|Z_i - Z_j| \tag{3.3}$$

for each time series of 500 terms of a given Z. The dimension m ranged from 1 to 16.

The moments where measured for three values of ε. The region of linearity between $\log C_m(\varepsilon)$ and $\log C_1(\varepsilon)$ or $\log\varepsilon$ varies with m, so we let the choice of ε depend on m by the following procedure. For each Z^k, we calculated 100 instances of $C_m(\varepsilon_i)$ and found the largest $\ell(m,j)$, for which

$$C_m(\varepsilon_{\ell(m,j)}) \le b_j, \qquad j = 1,2,3$$
$$\text{with } (b_1,b_2,b_3) = (0.05,\ 0.1,\ 0.15). \tag{3.4}$$

These numbers $\ell(m,j)$ where averaged over the 100 instances, obtaining $\bar{\ell}(m,j)$. With these preliminary numbers obtained, moments where now calculated for the random variables

$$CDR(m,\varepsilon_j) - m,\ CRL(m,\varepsilon_j),\ DBS(m,\varepsilon_j) \tag{3.5}$$

where $\varepsilon_j = \varepsilon_{(\bar{\ell}(m,j))}$, j = 1,2,3 (observe that to report the experiment we normalize the CDR so that its theoretical expected value, as ε tends to zero, is zero.

For each random variable Z^k (a series of 500 terms), 3000 instances were now calculated. From each of these, the random variables (3.5) were simultaneously obtained, obtaining therefore 3000 instances from which the moments and their significance were calculated.

We obtain the significance of skewness by the approximation of its null distribution by a Johnson S_U curve given by D'Agostino. For a sample of 3000 we have:

One-sided significance of Skewness

Significance level	0.10	0.05	0.025	0.01	0.005
Significance index	0.057	0.073	0.088	0.104	0.115

The significance of Kurtosis by the Anscombe and Glynn approximation, which yields an approximately normally distributed significance index:

One-sided significance of Kurtosis

Significance level	0.10	0.05	0.025	0.01	0.005	0.0025	0.001
Significance index	1.28	1.64	1.96	2.32	2.57	2.81	3.85

(D'Agostino and Stephens, 1986 is the reference for both significances).

5 The results

In Tables 1 to 6 we report for the processes Z^1 to Z^6 (respectively) the experimental value of μ/σ (mean divided by standard deviation) and the experimental values of skewness and kurtosis, and their significances.

On the one hand, in the IID cases Z^1 and Z^2, μ/σ should be as close to zero as possible. On the other, in the non-IID cases Z^3 to Z^6, the larger the experimental value of μ/σ, the more sensitively the statistic detects the dependent structure.

We also report the standard deviation of the CDR, CRL and DBS statistics in the following way. What is interesting is how heteroskedastic these distributions are as ε varies. Therefore we calculate the mean μ_σ and the standard deviation σ_σ of the three values $\sigma(\varepsilon_1)$, $\sigma(\varepsilon_2)$, $\sigma(\varepsilon_2)$, reporting $100\mu_\sigma$ (for reasons of scale) and σ_σ/μ_σ.

Our observations are the following.

(a) μ/σ. For the IID cases Z^1 and Z^2, the three statistics CDR, CRL, and DBS give similar magnitudes of deviation from the theoretical mean. The CDR has some much smaller deviations for m = 2, tending to increase with m, while the CRL and DBS deviations tend to decrease with m. In the case of non-IID distributions, however, the deviations are clearly much greater for the CDR, followed by the CRL, and then by the DBS statistic (in the case of Z^4 only a weak dependence structure is present, since the Markov process tends to alternate fairly regularly while its effect, doubling or halving the standard deviation, is not too strong). For these series the deviation tends to decrease with m for the three statistics.

(b) Skewness and Kurtosis. In both cases the CDR and the CRL (more the first than the second) are consistently better than the DBS statistic, with the improvement increasing with dimension. The CDR tends to be negatively skewed, with a positive mean, while the CRL and DBS statistics tend to be positively skewed, with a negative mean, (representing more probability for $C_m(\varepsilon) < C_1(\varepsilon)^m$ then vice-versa). The CDR is almost acceptably normal for many values of m in the cases Z^3 and Z^5 (ARCH) while its kurtosis is almost acceptably normal in the cases Z^1, Z^2 and Z^6. In general, the CDR and CRL, in that order, have less probability distributed in the tails than the DBS statistic. The DBS statistic becomes particularly distorted for non-IID random variables.

(c) Heteroskedasticity in ε. The values found for σ_σ/μ_σ are strikingly consistent and show that the sequence CDR, CRL, DBS is increasingly heteroskedastic (except in the case of the uniform random variable in which the CDR and the CRL change places for some intermediate values of m). For practical purposes the CDR may be said to have low or very low heteroskedasticity as ε varies.

It is worth mentioning that the CDR produces meaningful results for data sets well below the order $N = 10^m$ suggested to be necessary for dimension statistics by Ramsey, Sayers and Rothman (1990).

6 Conclusions

For time series of the order of 500 terms, the correlation dimension ratio (CDR), followed by the correlation ratio logarithm (CRL), are consistently more sensitive to stochastic dependence than the DBS statistic. Also, they are systematically closer (in the same order) to the normal distribution, having less skewness and kurtosis over a wide variety of dimensions and values of ε. Finally, they are less heteroskedastic (again in the same order) as ε varies over the range $0.05 \le C_m(\varepsilon) \le 0.15$, which for this length of time series represents the region where one may expect $C_m(\varepsilon)$ to be approximately log-linear with respect to $C_1(\varepsilon)$ or ε.

References.

BAEK, E. G. and W. A. BROCK (1992), "A General Test for Nonlinear Granger Causality: Bivariate Model." Preprint.

BROCK, W. A. (1986), "Distinguishing Random and Deterministic Systems: Abridged Version", Journal of Economic Theory 40 168-195.

BROCK, W. A., W. D. Dechert (1987), "Theorems on distinguishing deterministic from random systems. Social Systems Research Institute, University of Wisconsin (working paper number 8701).

BROCK, W. A., W. D. Dechert and J. A. Scheinkman, (1987), "A Test for Independence Based on the Correlation Dimension". Department of Economics, University of Wisconsin at Madison , University of Houston and University of Chicago.

BROCK, W. A., W. D. Dechert, J. A. Scheinkman and Blake LeBaron (Revised 1991), "A Test for Independence Based on the Correlation Dimension". Social Systems Research Institute, University of Wisconsin (working paper number 8702).

BROCK, W. A. and C. SAYERS (1988), "Is the business cycle characterized by deterministic chaos?", Journal of Monetary Economics 22, 71-90.

BROCK, W. A. and A. G. MALLIARIS (1989), Differential Equations, Stability and Chaos in Dynamic Economics. Advanced Textbooks in Economics, North-Holland.

BROCK, W. A. and E. G. BAEK (1991), "Some Theory of Statistical Inference for Nonlinear Science", Review of Economic Studies 58, 697-716.

D'AGOSTINO, R. B. and Michael A. Stephens (1986), editors, Goodness-of-fit techniques, Marcel Dekker, Inc, New York and Basle.

DENKER, M. and G. KELLER (1983), "On U-Statistics and V. Misses Statistics for Weakly Dependent Processes", Zeitschrift fur Wahrscheinlichkeitstheorie und Verwandte Gebiete, 64, 505-522.

GRASSBERGER, Peter and Itamar PROCACCIA (1983), "Measuring the strangeness of strange attractors", Physica 9D, 189-208.

MAYER, David (1995). "An Alternative Correlation Dimension Statistic", Journal of Empirical Finance, volume 2, issue 3.

PROVENZALE, A., L. A. Smith, R. Vio and G. Murante, "Distinguishing between low dimensional dynamics and randomness in measured time series, Physica D 58 (1992) 31-49.

RAMSEY, James B., Chera L. SAYERS and Philip ROTHMAN (1990) "The statistical Properties of Dimension Calculations using Small Data Sets: Some Economic Applications." International Economic Review, Vol 31, No 4, November 1990.

SERFLING, R.J. (1980), Approximation Theorems of Mathematical Statistics, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons.

THEILER, James (1990), "Estimating fractal dimension", Journal of the Optical Society of America, part A, Vol 7 No 6, pages 1055-1073.

m	statistic	μ/σ			Skewness			Kurtosis			Skewness (significance)			Kurtosis (significance)		
		ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3
2	CDR	0.08	0.07	0.08	-0.21	-0.26	-0.19	3.04	3.09	3.00	-4.60	-5.66	-4.14	0.54	1.00	0.11
	CRL	-0.21	-0.15	-0.13	0.21	0.27	0.20	3.04	3.02	2.88	4.65	6.04	4.36	0.52	0.28	-1.32
	DBS	-0.19	-0.14	-0.12	0.27	0.28	0.21	3.33	3.39	3.19	5.85	6.22	4.71	3.29	3.79	2.02
3	CDR	0.07	0.09	0.09	-0.17	-0.19	-0.12	3.10	3.06	3.00	-3.86	-4.23	-2.73	1.09	0.75	0.05
	CRL	-0.15	-0.13	-0.11	0.18	0.20	0.15	3.08	3.06	3.00	4.10	4.36	3.26	0.96	0.77	0.09
	DBS	-0.13	-0.11	-0.10	0.24	0.25	0.16	3.55	3.40	3.17	5.40	5.55	3.53	4.97	3.81	1.81
4	CDR	0.09	0.09	0.08	-0.23	-0.17	-0.12	3.21	3.14	3.04	-5.08	-3.69	-2.71	2.22	1.50	0.50
	CRL	-0.15	-0.12	-0.10	0.24	0.20	0.15	3.27	3.20	3.13	5.28	4.43	3.31	2.77	2.09	1.43
	DBS	-0.12	-0.10	-0.09	0.33	0.21	0.17	3.82	3.53	3.21	7.12	4.73	3.86	6.72	4.84	2.19
5	CDR	0.09	0.09	0.09	-0.20	-0.17	-0.12	3.13	3.08	3.04	-4.34	-3.81	-2.77	1.41	0.90	0.48
	CRL	-0.13	-0.11	-0.10	0.23	0.20	0.15	3.20	3.20	3.22	5.04	4.48	3.40	2.08	2.11	2.30
	DBS	-0.10	-0.10	-0.09	0.26	0.24	0.19	3.88	3.48	3.21	5.75	5.29	4.20	7.08	4.43	2.18
6	CDR	0.10	0.09	0.09	-0.19	-0.16	-0.14	3.09	3.03	3.05	-4.20	-3.65	-3.03	1.05	0.42	0.60
	CRL	-0.13	-0.11	-0.11	0.22	0.21	0.17	3.22	3.20	3.32	4.89	4.59	3.86	2.28	2.14	3.13
	DBS	-0.10	-0.09	-0.09	0.27	0.23	0.21	3.82	3.50	3.21	5.91	5.18	4.62	6.70	4.61	2.22
8	CDR	0.11	0.11	0.10	-0.19	-0.16	-0.15	2.96	2.99	3.00	-4.26	-3.54	-3.30	-0.40	-0.03	0.02
	CRL	-0.13	-0.12	-0.11	0.22	0.17	0.16	3.12	3.24	3.38	4.80	3.87	3.50	1.35	2.43	3.68
	DBS	-0.09	-0.09	-0.09	0.33	0.28	0.25	4.03	3.51	3.21	7.27	6.21	5.55	7.90	4.69	2.17
16	CDR	0.16	0.15	0.14	-0.20	-0.19	-0.20	3.04	3.11	3.07	-4.54	-4.14	-4.40	0.48	1.19	0.87
	CRL	-0.18	-0.16	-0.15	0.11	0.09	0.05	3.38	3.47	3.67	2.43	1.91	1.10	3.70	4.37	5.77
	DBS	-0.10	-0.11	-0.11	0.58	0.49	0.40	5.42	4.43	3.59	12.19	10.42	8.73	13.21	9.77	5.24

Table 1. Comparison of the CDR, CRL and DBS satistics for Z^i, a normally distributed random variable N(0,1).

m	statistic	μ/σ			Skewness			Kurtosis			Skewness (significance)			Kurtosis (significance)		
		ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3
2	CDR	0.02	0.02	0.05	-0.43	-0.61	-0.41	3.51	4.00	3.75	-9.20	-12.61	-8.90	4.69	7.71	6.31
	CRL	-0.21	-0.17	-0.15	0.43	0.61	0.41	3.52	3.98	3.72	9.28	12.57	8.88	4.72	7.62	6.08
	DBS	-0.21	-0.17	-0.15	0.46	0.64	0.43	3.58	4.10	3.82	9.74	13.15	9.30	5.14	8.26	6.73
3	CDR	0.03	0.05	0.06	-0.39	-0.35	-0.28	3.53	3.26	3.30	-8.53	-7.52	-6.24	4.80	2.69	3.04
	CRL	-0.17	-0.14	-0.11	0.39	0.34	0.28	3.51	3.23	3.25	8.53	7.47	6.24	4.65	2.37	2.59
	DBS	-0.16	-0.13	-0.10	0.45	0.39	0.32	3.70	3.39	3.39	9.60	8.40	6.94	5.98	3.75	3.74
4	CDR	0.06	0.06	0.07	-0.32	-0.34	-0.20	3.20	3.24	3.00	-7.07	-7.50	-4.45	2.13	2.49	0.05
	CRL	-0.15	-0.12	-0.10	0.32	0.34	0.20	3.17	3.20	2.97	6.98	7.43	4.45	1.85	2.09	-0.30
	DBS	-0.14	-0.11	-0.09	0.40	0.41	0.25	3.40	3.42	3.08	8.65	8.79	5.45	3.82	3.95	0.90
5	CDR	0.06	0.06	0.08	-0.32	-0.30	-0.18	3.21	3.18	3.02	-7.10	-6.66	-3.96	2.23	1.95	0.32
	CRL	-0.13	-0.11	-0.10	0.32	0.30	0.17	3.18	3.13	3.01	6.97	6.55	3.79	1.91	1.43	0.13
	DBS	-0.11	-0.09	-0.09	0.43	0.39	0.25	3.48	3.41	3.11	9.24	8.43	5.43	4.41	3.91	1.23
6	CDR	0.07	0.08	0.09	-0.28	-0.22	-0.15	3.07	3.00	3.02	-6.14	-4.78	-3.36	0.88	0.10	0.28
	CRL	-0.13	-0.11	-0.11	0.27	0.21	0.14	3.03	2.99	3.01	6.04	4.69	3.14	0.36	-0.10	0.19
	DBS	-0.11	-0.09	-0.10	0.40	0.31	0.23	3.41	3.19	3.12	8.67	6.85	5.19	3.94	2.06	1.38
8	CDR	0.09	0.11	0.10	-0.25	-0.17	-0.20	3.12	3.08	3.01	-5.61	-3.74	-4.34	1.36	0.96	0.14
	CRL	-0.13	-0.13	-0.12	0.26	0.16	0.19	3.10	3.07	3.03	5.68	3.66	4.18	1.14	0.80	0.42
	DBS	-0.11	-0.10	-0.10	0.41	0.30	0.30	3.55	3.36	3.16	8.80	6.67	6.49	4.98	3.53	1.78
16	CDR	0.15	0.13	0.11	-0.31	-0.31	-0.27	3.29	3.27	3.16	-6.76	-6.70	-5.87	2.89	2.72	1.71
	CRL	-0.17	-0.15	-0.12	0.30	0.29	0.24	3.39	3.40	3.33	6.64	6.46	5.32	3.73	3.83	3.28
	DBS	-0.12	-0.11	-0.09	0.62	0.55	0.44	4.16	3.83	3.39	12.87	11.63	9.35	8.52	6.78	3.77

Table 2. Comparison of the CDR, CRL and DBS satistics for Z^i, a uniformly distributed random variable.

m	statistic	μ/σ			Skewness			Kurtosis			Skewness (significance)			Kurtosis (significance)		
		ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3
2	CDR	-9.01	-9.37	-9.13	-0.05	-0.07	-0.05	2.90	2.93	2.93	-1.08	-1.59	-1.07	-1.16	-0.69	-0.71
	CRL	7.67	6.85	5.67	0.18	0.17	0.11	3.05	2.98	2.91	4.08	3.72	2.48	0.57	-0.14	-0.93
	DBS	3.73	4.68	6.31	0.66	0.41	0.15	3.75	3.23	2.88	13.58	8.75	3.32	6.28	2.34	-1.37
3	CDR	-6.58	-6.49	-6.01	-0.11	-0.04	-0.02	2.83	2.86	3.06	-2.46	-0.79	-0.38	-1.97	-1.66	0.69
	CRL	5.39	4.88	4.05	0.26	0.21	0.25	3.03	2.95	2.99	5.72	4.57	5.46	0.38	-0.48	-0.05
	DBS	2.81	3.45	4.51	0.90	0.58	0.23	4.42	3.43	2.91	17.47	12.11	5.13	9.74	4.09	-0.94
4	CDR	-5.50	-5.38	-4.95	-0.01	0.03	0.01	2.88	2.99	3.11	-0.33	0.59	0.12	-1.37	-0.07	1.28
	CRL	4.34	3.98	3.40	0.26	0.26	0.34	3.05	3.05	3.08	5.72	5.83	7.44	0.62	0.67	0.93
	DBS	2.50	3.00	3.74	0.91	0.56	0.25	4.24	3.25	3.01	17.54	11.66	5.45	8.92	2.61	0.13
5	CDR	-4.78	-4.62	-4.15	0.03	0.03	0.01	2.93	3.09	3.08	0.70	0.74	0.28	-0.73	0.99	0.95
	CRL	3.79	3.44	2.92	0.28	0.33	0.41	3.10	3.13	3.13	6.25	7.17	8.85	1.09	1.44	1.45
	DBS	2.15	2.64	3.23	1.00	0.62	0.33	4.38	3.37	3.15	18.96	12.75	7.18	9.55	3.61	1.67
6	CDR	-4.26	-4.04	-3.61	0.03	0.02	0.01	2.99	3.09	3.05	0.76	0.49	0.13	-0.08	0.99	0.66
	CRL	3.38	3.02	2.60	0.31	0.39	0.47	3.15	3.17	3.19	6.87	8.38	10.01	1.59	1.85	1.98
	DBS	1.94	2.42	2.84	1.08	0.66	0.40	4.54	3.50	3.26	20.05	13.53	8.69	10.22	4.59	2.62
8	CDR	-3.43	-3.23	-2.90	0.03	0.03	-0.01	3.00	3.00	2.92	0.59	0.68	-0.22	0.04	0.10	-0.88
	CRL	2.73	2.49	2.18	0.42	0.47	0.57	3.25	3.24	3.29	9.13	10.04	11.90	2.56	2.46	2.91
	DBS	1.69	2.03	2.31	1.17	0.79	0.51	4.91	3.78	3.34	21.23	15.75	10.77	11.63	6.47	3.31
16	CDR	-1.94	-1.83	-1.63	0.07	0.09	0.04	2.98	2.99	3.08	1.66	2.03	0.92	-0.11	-0.04	0.95
	CRL	1.63	1.53	1.36	0.54	0.57	0.70	3.56	3.56	3.86	11.27	11.95	14.18	5.03	5.03	6.95
	DBS	1.16	1.31	1.39	1.47	1.03	0.66	6.02	4.44	3.81	24.76	19.38	13.47	14.77	9.81	6.65

Table 3. Comparison of the CDR, CRL and DBS satistics for Z^3, a random variable $N(0,|\sigma|)$ with $\sigma \in N(0,1)$.

m	statistic	μ/σ			Skewness			Kurtosis			Skewness (significance)			Kurtosis (significance)		
		ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3
2	CDR	-1.09	-1.19	-1.28	-0.16	-0.25	-0.18	3.37	3.28	3.20	-3.66	-5.51	-3.96	3.60	2.81	2.13
	CRL	0.99	1.14	1.25	0.14	0.23	0.17	3.37	3.28	3.24	3.18	5.16	3.74	3.58	2.87	2.50
	DBS	0.96	1.11	1.22	0.44	0.45	0.30	3.91	3.66	3.37	9.45	9.71	6.66	7.21	5.70	3.57
3	CDR	-1.25	-1.33	-1.41	-0.30	-0.25	-0.20	3.32	3.21	3.12	-6.53	-5.50	-4.45	3.19	2.23	1.37
	CRL	1.20	1.30	1.38	0.27	0.25	0.25	3.37	3.34	3.36	5.98	5.46	5.50	3.56	3.31	3.54
	DBS	1.12	1.23	1.35	0.76	0.58	0.37	4.34	3.70	3.27	15.26	12.02	8.11	9.37	5.96	2.76
4	CDR	-1.26	-1.34	-1.41	-0.29	-0.25	-0.21	3.25	3.12	3.10	-6.36	-5.43	-4.71	2.58	1.37	1.18
	CRL	1.22	1.31	1.37	0.27	0.27	0.30	3.35	3.30	3.43	5.90	5.85	6.63	3.40	3.00	4.03
	DBS	1.11	1.22	1.34	0.89	0.65	0.43	4.50	3.66	3.30	17.23	13.29	9.15	10.07	5.73	3.03
5	CDR	-1.22	-1.29	-1.34	-0.24	-0.22	-0.20	3.09	3.08	3.10	-5.34	-4.85	-4.44	1.07	0.93	1.12
	CRL	1.18	1.25	1.30	0.25	0.27	0.32	3.23	3.31	3.44	5.46	5.95	7.03	2.42	3.12	4.12
	DBS	1.06	1.16	1.27	0.90	0.69	0.45	4.31	3.75	3.37	17.45	14.04	9.72	9.23	6.26	3.61
6	CDR	-1.17	-1.23	-1.27	-0.22	-0.19	-0.19	3.09	3.04	3.06	-4.78	-4.27	-4.32	1.05	0.56	0.67
	CRL	1.14	1.19	1.22	0.25	0.28	0.37	3.30	3.35	3.52	5.42	6.26	7.97	2.99	3.45	4.71
	DBS	1.01	1.12	1.21	0.97	0.69	0.45	4.56	3.77	3.33	18.42	14.02	9.67	10.29	6.42	3.30
8	CDR	-1.07	-1.12	-1.12	-0.20	-0.17	-0.19	3.01	2.99	3.02	-4.34	-3.89	-4.16	0.19	-0.02	0.25
	CRL	1.03	1.08	1.07	0.26	0.30	0.42	3.29	3.37	3.61	5.84	6.49	8.97	2.96	3.59	5.37
	DBS	0.92	1.01	1.07	1.07	0.74	0.47	4.91	3.89	3.34	19.82	14.88	9.99	11.60	7.11	3.38
16	CDR	-0.72	-0.71	-0.65	-0.03	-0.05	-0.11	2.99	3.11	3.24	-0.72	-1.12	-2.35	-0.07	1.22	2.51
	CRL	0.68	0.67	0.62	0.18	0.27	0.41	3.58	3.90	4.43	4.06	6.00	8.77	5.19	7.18	9.76
	DBS	0.65	0.67	0.65	1.06	0.70	0.45	5.53	4.40	3.73	19.70	14.23	9.73	13.52	9.64	6.18

Table 4. Comparison of the CDR, CRL and DBS satistics for Z^t, a random variable $N(0,\sigma)$ with σ following a Markov process.

m	statistic	μ/σ			Skewness			Kurtosis			Skewness (significance)			Kurtosis (significance)		
		ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3
2	CDR	-8.01	-8.89	-9.57	-0.12	-0.09	-0.06	3.02	2.95	2.85	-2.76	-1.96	-1.30	0.27	-0.54	-1.79
	CRL	8.22	7.91	6.81	0.08	0.07	-0.00	3.01	2.95	2.97	1.84	1.47	-0.01	0.18	-0.53	-0.24
	DBS	3.14	3.97	5.37	0.93	0.59	0.25	4.36	3.38	2.82	17.95	12.32	5.56	9.47	3.63	-2.15
3	CDR	-10.15	-10.42	-10.34	-0.08	-0.08	-0.05	2.97	2.83	3.08	-1.80	-1.75	-1.10	-0.24	-2.07	0.89
	CRL	7.71	6.72	5.40	0.04	0.02	0.03	2.86	2.88	2.84	0.79	0.36	0.61	-1.60	-1.42	-1.88
	DBS	2.79	3.46	5.10	0.99	0.65	0.24	4.54	3.49	2.90	18.78	13.40	5.28	10.22	4.53	-1.13
4	CDR	-9.74	-9.86	-9.46	-0.04	-0.03	-0.02	3.00	2.92	2.97	-0.94	-0.78	-0.45	0.11	-0.89	-0.25
	CRL	6.49	5.74	4.66	0.06	0.06	0.09	2.83	2.86	2.83	1.38	1.25	2.12	-2.03	-1.67	-1.99
	DBS	2.45	3.01	4.54	1.04	0.72	0.25	4.58	3.62	2.89	19.51	14.59	5.54	10.41	5.41	-1.18
5	CDR	-9.27	-9.19	-8.70	0.00	0.01	0.01	2.94	3.07	2.99	0.02	0.21	0.12	-0.63	0.86	-0.10
	CRL	5.77	5.03	4.20	0.09	0.09	0.15	2.87	2.83	2.85	1.99	2.06	3.43	-1.55	-2.01	-1.72
	DBS	2.17	2.77	4.08	1.15	0.73	0.29	4.84	3.57	2.91	20.88	14.70	6.33	11.38	5.08	-1.02
6	CDR	-8.71	-8.59	-8.07	0.03	0.02	0.03	2.89	2.87	3.04	0.60	0.55	0.61	-1.27	-1.54	0.50
	CRL	5.20	4.61	3.90	0.12	0.13	0.21	2.84	2.83	2.89	2.60	2.89	4.70	-1.86	-2.08	-1.24
	DBS	2.00	2.53	3.67	1.20	0.78	0.33	4.95	3.62	2.92	21.56	15.52	7.30	11.76	5.45	-0.82
8	CDR	-7.73	-7.52	-7.03	0.06	0.05	0.01	2.84	2.89	2.94	1.29	1.22	0.15	-1.89	-1.21	-0.67
	CRL	4.50	3.96	3.43	0.15	0.21	0.33	2.80	2.86	3.00	3.31	4.71	7.29	-2.39	-1.58	0.06
	DBS	1.74	2.28	3.23	1.28	0.81	0.40	5.08	3.61	2.95	22.62	15.96	8.65	12.18	5.36	-0.56
16	CDR	-5.35	-5.08	-4.66	0.12	0.07	0.00	2.90	2.94	2.96	2.62	1.48	0.05	-1.12	-0.62	-0.34
	CRL	3.18	2.85	2.55	0.37	0.51	0.60	3.04	3.22	3.37	8.10	10.71	12.53	0.55	2.33	3.57
	DBS	1.35	1.78	2.43	1.48	0.97	0.59	5.81	4.02	3.33	24.85	18.41	12.38	14.26	7.83	3.26

Table 5. Comparison of the CDR, CRL and DBS satistics for Z^3, an ARCH process as described in the text.

m	statistic	μ/σ			Skewness			Kurtosis			Skewness (significance)			Kurtosis (significance)		
		ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3
2	CDR	-6.45	-6.69	-7.13	-0.20	-0.19	-0.14	2.97	3.05	3.01	-4.54	-4.30	-3.13	-0.32	0.60	0.18
	CRL	6.64	6.87	6.76	0.20	0.20	0.23	3.06	3.13	3.09	4.35	4.45	5.13	0.77	1.48	1.06
	DBS	3.81	4.20	5.22	0.76	0.64	0.35	4.09	3.72	3.10	15.31	13.19	7.54	8.21	6.07	1.12
3	CDR	-6.71	-7.03	-7.56	-0.19	-0.15	-0.11	3.00	2.97	2.97	-4.31	-3.30	-2.43	0.09	-0.26	-0.32
	CRL	6.79	6.69	6.13	0.22	0.24	0.23	3.08	3.06	3.05	4.79	5.20	5.18	0.95	0.77	0.60
	DBS	3.14	3.68	4.88	0.95	0.68	0.33	4.64	3.63	3.01	18.17	13.81	7.14	10.64	5.48	0.13
4	CDR	-6.85	-7.25	-7.66	-0.14	-0.12	-0.09	2.96	2.94	2.98	-3.16	-2.79	-2.00	-0.34	-0.64	-0.14
	CRL	6.55	6.28	5.54	0.22	0.24	0.21	3.03	3.06	3.01	4.81	5.24	4.60	0.38	0.70	0.23
	DBS	2.73	3.29	4.47	1.02	0.70	0.34	4.68	3.57	3.00	19.17	14.27	7.44	10.76	5.09	0.10
5	CDR	-6.98	-7.33	-7.58	-0.13	-0.10	-0.08	2.95	2.93	3.00	-2.83	-2.25	-1.82	-0.54	-0.73	0.09
	CRL	6.19	5.80	4.96	0.25	0.23	0.21	2.99	3.01	2.97	5.43	4.99	4.61	0.01	0.22	-0.30
	DBS	2.49	3.04	4.32	1.04	0.71	0.32	4.58	3.54	3.02	19.49	14.44	7.06	10.41	4.87	0.30
6	CDR	-7.00	-7.30	-7.40	-0.10	-0.08	-0.08	2.97	2.99	3.00	-2.26	-1.82	-1.82	-0.30	0.01	0.01
	CRL	5.83	5.30	4.59	0.24	0.22	0.22	2.99	2.97	2.93	5.23	4.86	4.91	-0.08	-0.30	-0.77
	DBS	2.30	2.93	4.03	1.08	0.69	0.35	4.61	3.49	3.07	19.95	14.14	7.69	10.52	4.51	0.81
8	CDR	-6.94	-7.06	-6.92	-0.08	-0.08	-0.06	2.99	3.03	3.02	-1.73	-1.75	-1.37	-0.04	0.41	0.25
	CRL	5.17	4.64	3.95	0.22	0.23	0.28	2.93	2.92	2.94	4.88	5.16	6.17	-0.76	-0.92	-0.66
	DBS	2.07	2.65	3.81	1.13	0.74	0.37	4.73	3.59	3.18	20.68	14.87	8.01	10.97	5.25	1.93
16	CDR	-5.84	-5.62	-5.17	-0.02	-0.04	-0.04	2.99	3.00	3.11	-0.52	-0.85	-0.79	0.00	0.09	1.20
	CRL	3.67	3.31	2.89	0.31	0.38	0.48	2.83	2.93	3.08	6.82	8.31	10.22	-1.97	-0.72	0.94
	DBS	1.65	2.13	3.00	1.30	0.89	0.52	5.19	3.95	3.39	22.79	17.22	11.04	12.52	7.46	3.74

Table 6. Comparison of the CDR, CRL and DBS satistics for Z^4, an autoregressive change of state process as described in the text.

m	statistic	Stochastic process											
		Z^1		Z^2		Z^3		Z^4		Z^5		Z^6	
		$100\ \mu_\sigma$	σ_σ/μ_σ	$100\ \mu_\sigma$	σ_σ/μ_σ	$100\ \mu_\sigma$	σ_σ/μ_σ	$100\ \mu_\sigma$	σ_σ/μ_σ	$100\ \mu_\sigma$	σ_σ/μ_σ	$100\ \mu_\sigma$	σ_σ/μ_σ
2	CDR	1.19	0.10	0.62	0.06	3.05	0.02	1.65	0.13	3.52	0.00	2.82	0.11
	CRL	1.33	0.16	0.72	0.30	5.55	0.14	1.85	0.13	5.44	0.10	3.43	0.10
	DBS	0.15	0.39	0.07	0.28	0.63	0.12	0.21	0.41	0.84	0.23	0.51	0.34
3	CDR	3.10	0.08	1.57	0.12	7.53	0.01	4.48	0.09	7.03	0.01	6.60	0.05
	CRL	2.32	0.17	1.17	0.14	9.36	0.18	3.39	0.15	11.05	0.09	6.17	0.09
	DBS	0.25	0.35	0.13	0.41	0.94	0.16	0.39	0.35	1.38	0.17	0.95	0.24
4	CDR	5.71	0.06	3.23	0.17	12.80	0.01	8.21	0.06	11.51	0.02	10.68	0.02
	CRL	3.24	0.19	1.77	0.10	12.03	0.17	4.81	0.18	15.69	0.11	8.69	0.09
	DBS	0.34	0.34	0.20	0.46	1.20	0.12	0.53	0.33	1.80	0.14	1.27	0.20
5	CDR	8.91	0.04	5.50	0.16	18.93	0.01	12.75	0.04	16.54	0.02	15.01	0.02
	CRL	4.03	0.20	2.41	0.09	14.40	0.20	6.10	0.19	19.69	0.13	10.99	0.09
	DBS	0.43	0.31	0.27	0.44	1.37	0.15	0.66	0.28	2.11	0.14	1.53	0.17
6	CDR	12.60	0.02	8.40	0.14	25.77	0.02	17.88	0.02	22.14	0.02	19.60	0.01
	CRL	4.83	0.22	3.09	0.11	16.26	0.21	7.23	0.21	23.23	0.14	13.13	0.10
	DBS	0.50	0.29	0.35	0.41	1.51	0.16	0.78	0.27	2.37	0.14	1.75	0.16
8	CDR	21.11	0.01	15.42	0.07	41.62	0.01	29.37	0.01	34.94	0.02	29.52	0.02
	CRL	6.11	0.25	4.32	0.17	19.45	0.21	9.20	0.22	28.60	0.15	16.79	0.12
	DBS	0.63	0.28	0.47	0.35	1.75	0.16	0.95	0.25	2.77	0.14	2.06	0.13
16	CDR	63.62	0.02	50.56	0.02	123.71	0.00	89.65	0.01	103.71	0.03	80.31	0.03
	CRL	9.41	0.26	7.39	0.26	26.85	0.23	14.60	0.24	40.48	0.19	26.55	0.17
	DBS	0.96	0.23	0.74	0.27	2.34	0.16	1.40	0.22	3.70	0.11	2.78	0.09

Table 7. Mean, and standard deviation divided by the mean, of the standard deviations $\sigma(\varepsilon_1)$, $\sigma(\varepsilon_2)$ and $\sigma(\varepsilon_3)$ obtained at three values of ε for the CDR, CRL and DBS satistics of each of the stochastic processes Z^i.