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MATCHING PROBLEM IN A MULTI-VARIATE SETTING

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PRESENTA

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Dedíco este trabajo a:
Mi familia,
mis compañeros y
al CIDE.

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y especialmente a mis padres.

Abstract

In this work we propose a model to study the matching problem in a setting where agent characteristics are multi-dimensional. We draw a parallel between matching problems and the adverse selection literature in order to characterize feasible matches with the incentive compatibility constraint. Our model also provide a natural ordering of the type spaces that is consistent with our theory, we use this order to find necessary conditions for positive assortative matching PAM.

Contenido

Chapter 1

Introduction

Matching problems are of increasing interest in the economic literature from labor markets and industrial organization, (Koopmans and Beckmann, 1957) to household and consumer behavior (Shapley and Shubik, 1971), or even a theory on marriage, (Becker, 1973). Most of this literature focuses on situations where a single parameter captures the characteristics of matching partners. For example, in a labor market, employers who try to find the most capable workers and workers want the best possible job for themselves. In this example, worker productivity and job desirability are real numbers which we can compare. Another example, two sets of firms need to pair in order to produce, firms on both sides of the market have different levels of productivity that affects the final output.

However, representing characteristics with a single parameter is often an oversimplification as (Rochet and Stole, 2003, p. 150) put it, "... in most cases that we can think of, a multidimensional preference parameterization seems critical to capturing the basic economics of the environment." On the one hand, taking into account, various characteristics enrich the understanding of the problem, but on the other hand, adding just one more reduces tractability in a significant matter. Notably, there is no equivalent for monotonicity in a multidimensional setting.

When characteristics are unidimensional, then we can draw a parallel with the adverse selection literature. It is advantageous to usencentive Compatibility and Individual Rationality constraints to characterize conditions under which a match would occur as I. These restrictions allow us to be specific on the properties of optimal matching rules, precisely stability, and assortativity. In this work, we propose to use projections of characteristics on to the real line to use the tools of mechanism design in order to define assortativity.

This work has the following structure: the following section presents a summary on the literature of multidimensional screening and multidimensional matching. In *Chapter 3* first we establish the parallel between a matching model where utilities are transferable and a standard monopolistic screening model; then we propose a multidimensional generalization of that model and conclude the properties of the optimal matching when types are multidimensional. *Chapter 4* is an application of our results to a signaling game. In *Chapter 5*, we explore the case when there are utilities are non-transferable, and in *Chapter 6*, we present our conclusions.

1.1 Literature Review

The literature of optimal screening contract goes back to the 1970s. (Rothschild and Stiglitz, 1978) propose a model to deal with asymmetric information present in insurance markets. (Mussa and Rosen, 1978) in their seminal paper develop the theory for a monopoly that implements a nonlinear pricing scheme $P(q)$ depending on the quantity q of the product that the consumer chooses, the population of potential consumers have preferences fully characterized by a parameter θ. The parameter θ is private information of the costumer and the monopolist only know it is cumulative distribution function $F(\theta)$.

In the kind of problems that attain us, there is private information or actions that cannot be verified; then incentive constraints must be taken in to account. The optimization problem no longer solely determined by each agent's rationality but also by the externalities created by private information. The methodology for analyzing these constraints is the theory of *mechanism design*, quoting (Myerson, 1988): "A mechanism is a specification of how economic decisions are determined as a function of the information that is known by the individuals in the economy." If every individual in the economy reports their private information, then any rule that determines how the economy will behave as a function of the information received is called a *direct mechanism*. If a direct mechanism is such that no participant will be better off by lying about her type, then the mechanism is said to be *incentive compatible*.

The importance of incentive compatible direct mechanism becomes central in mechanism theory from the result know as *revelation principle*, first recognized by (Gibbard, 1973) in the context of dominate strategies and generalized later by (Myerson, 1979), (Dasgupta, Hammond, and Maskin, 1979) and (Myerson, 1982) for the more broader solution concept of Bayesian equilibrium. The revelation principle states that (quoting): "...for any equilibrium of any general mechanism, there is an incentive compatible direct revelation mechanism that is essentially equivalent..." (Myerson, 1988, p. 2).

However, the primary concern of this work is to analyze how matching will occur when individual characteristics differ in more than one dimension. For example, consider the following reformulation of the labor market problem as in (Lindenlaub, 2017) were workers are endowed with cognitive and manual skills and employers with cognitive and manual skills demands. In order to describe these matching rules with mechanism design tools, we have to propose a multi-dimensional screening model. There is an obvious problem while studying matching assortativity in multi-dimensional settings: there is no complete ordering in a multi-dimensional space; so we will need to rethink the meaning of assortativity in this context.

The first instance of multi-dimensional screening in the literature is (Mirrlees, 1976) expanded in (Mirrlees, 1986) where the author analyzes preferences and products with more than one dimension as a possible extension to a taxation model, the author considers a *n-dimensional* product space, with a *m-dimensional* parameter space characterizing the consumer preferences. Provided that $m < n$ he provides first order conditions an incentive scheme to be optimal.

(Spence, 1980) explores products with more than one dimension. Spence considers a model where consumers are arranged in n groups and each group i purchase a bundle of m products $x_i = (x_{i1}, ..., x_{im})$, priced as one according to some pricing scheme $p = (p_1, ..., p_n)$. Since in this model the author does not impose any structure on the groups of consumers, it is not easy to draw a parallel between those groups and any characterization of the consumer preferences in terms of parameters. The author remarks that in the case of a single product there is a natural order of the agents by their utility function; such an order is not possible when dealing with multi-dimensional product spaces. (Maskin et al., 1987), consider the problem of a single good monopolist facing a continuum of buyers characterized by the pair $(\theta_1, \theta_2) \in [0, 1]^2$, the authors solve the model for a specific utility function.

(Armstrong, 1996) discusses the problems that arise when working in higher dimensions both of the product and the type spaces; he develops a model using additive preferences and concludes that it is optimal for the principal to exclude some consumers. (Rochet and Choné, 1998) generalizes to the results of (Mussa and Rosen, 1978) to multiple dimensions, also applicable to the formulations of (Armstrong, 1996). They argue that for the non-excluded costumers some of them who have different tastes will purchase the same combination of products, a phenom called “bunching”.

An essential property many results rely upon is *single crossing*. In words, this property guarantees that two indifference curves “cross” each other at most one time. By virtue of this property if a consumer with a lower type (i.e., a lower valuation of goods) is does not prefer (x, p) to (x', p') (consumption and price), where $x < x'$, then another consumer with a higher type will be willing to pay more than p' to acquire x'. Note that single crossing is a property of the utility function of the consumer and does not depend on the information structure present in the economy. (Milgrom and Shannon, 1994) relate the single crossing condition as an ordinal property with the *Spence-Mirrlees* which is a differential one. Since the result that the authors obtain is a necessary and sufficient condition for single crossing this implies that the problem of whether a direct mechanism is implementable reduces to checking properties of the derivatives of smooth functions.

While the result of (Milgrom and Shannon, 1994) “solves” the problem for uni-dimensional goods and types with smooth utility functions; Spence-Mirrlees is not useful in a context where those assumptions do not hold. (McAfee and McMillan, 1988) develop a model with m types $\theta = (\theta_1, .., \theta_m)$, and n products $x = (x_1, ..., x_n)$, with the restriction that $n \leq m$. They propose a generalized Spence-Mirrlees condition (GSM) and show that any mechanism that satisfies the first and second order conditions of the agent's maximization problem will be incentive compatible. (Rochet, 2009) and (Araujo and Moreira, 2010) analyze adverse selection problems when the assumption of the Spence-Mirrless condition is no longer valid, but they do so for uni-dimensional problems.

There is a vast body of literature covering matching models of uni-dimensional types; for a comprehensive survey on this topic see (Chade, Eeckhout, and Smith, 2017). The most significant result in the early literature on the subject was obtained by (Becker, 1973) in his seminal paper about marriages. He proves that the monotonicity of the matching will depend on the complementary of the partner's characteristics. More formally if the marriage “production function” is supermodular (submodular) then the optimal matching will pair higher types with higher(lower) types.

The topic of multi-dimensional matching is much more underdeveloped. However there are two recent works (Chiappori, McCann, and Pass, 2016) and (Lindenlaub, 2017) of significant importance. (Chiappori et al., 2016) states that due to the complexity o multi-dimensional choice sets there is no closed form solution for the multi-dimensional model. Even with these shortcomings, they do provide a characterization in terms of existence and uniqueness of the solution as well as qualitative properties of stable matches.

(Lindenlaub, 2017) propose a labor market model where employers and employees differ in two characteristics. The author points out that collapsing various characteristics into a single index leads to tractability. However, there are situations where there is no correlation of characteristics where such a projection would make sense.

An earlier work that uses projection of multi-dimensional types in the real line is (Mookherjee and Reichelstein, 1992). However in their case they use projections to define a *condensation property* of the cost functions. In the present work, we propose to use such projections over the type-spaces o both sides in order to endow these sets with an order that is consistent with our theory. We are proposing a matching model for multi-dimensional types based on two features:

- Pay-off functions for both agents to represent the problem as a screening one.
- Projections of the type-spaces on to the real line effectively collapsing various characteristics into one allow us to establish monotonicity of matching through an ordering implied by these projections.

Chapter 2

Model

2.1 Matching Model with Transferable Utilities TU

Consider a matching model as in (Legros and Newman, 2007): an economy populated by a continuum of agents distinguished by their type. We can consider that types are draw from two sets Q and Θ, this is consistent with a *two-sided* model, in which agents from each side need to "match" in order to produce. Consider the payoffs obtained in autarchy for all types to be normalized to 0.

Consider the utility possibility frontier $\phi : Q \times \Theta \times \mathbb{R}_+ \to \mathbb{R}_+$ where $\phi(q, \theta, u)$ is the maximum utility obtained by a type $q \in Q$ in a match with a type $\theta \in \Theta$ which obtains utility u. We can also consider the "inverse" of $\phi(q, \theta, \cdot)$ as $\psi(\theta, q, \cdot)$ then the following holds:

- $\phi(q, \theta, 0)$ is the maximum equilibrium payoff of type q.
- $\phi(q, \theta, \psi(\theta, q, v)) = v$ if $v \leq \phi(q, \theta, 0)$ and 0 otherwise.

We will say that we are in presence of transferable utilities TU when there is a production function $f(q, \theta)$ such that the frontier can be written:

$$\phi(q,\theta,u) = \begin{cases} f(q,\theta) - u \text{ if } u \in [0, f(q,\theta)] \\ 0 \qquad \text{if } u > f(q,\theta) \end{cases}$$

We will evaluate matches under two properties *stability* and *assortativity*, intuitively they mean:

- *Stability:* In equilibrium does not exist any match by which both agents matched would be individually better off than they are with the element to which they are currently matched.

- *Assortativity:* Does the matching rule preserve the order of the type-sets? In particular we are interested in matching patterns that are satisfy *positive assortative matching* henceforth *PAM*, those are monotone in the sense that pair "higher" types with "higher" types on the other side of the market.

Our objective first to establish a parallel of matching model with a particular screening model when types are uni-dimensional then use a generalization of this model to extend the results to a situation where types are drawn from a multi-dimensional type space.

2.2 One-Dimensional Model

In this section we will consider the types as real-valued, i.e. type spaces $Q \times \Theta \subset \mathbb{R}^2$. Now recall the (Maskin and Riley, 1984) model on monopoly with imperfect information: consider a monopoly that produces a single good $q \in Q \subset \mathbb{R}$ and agents are characterized by their type $\theta \in \Theta$ that constitutes their private information. Let $W(q,t) = t - cq$ and $U(q,t,\theta) = \theta V(q) - t$ represents the utility of the principal and the agent respectively. The agent has an outside option of U_0 which we will consider to be equal to 0. It is assumed that $V(\cdot)$ is increasing and strictly concave in q.

By the revelation principle, the principal can implement a direct revelation mechanism where the agents reveal their type and then they are assigned a consumption-transfer pair $(q(\theta), t(\theta))$. Then the principal's problem, is to design a mechanism that maximizes his expected payoff taking in to account the following constrains:

$$\theta V(q(\theta)) - t(\theta) \geq 0 \tag{IR_θ}$$

$$\theta \in \arg\max_{\tilde{\theta}} \{\theta V(q(\tilde{\theta})) - t(\tilde{\theta})\} \tag{IC_θ}$$

(IR_θ) is the participation constraint that captures that the agent has to get at least his reservation utility for the transaction to take place. The second one (IC_θ) is the incentive compatibility constraint and captures that the agent cannot be better-off by revealing a type different than her true type.

How does this formulation relates with the matching situation? Instead of a monopoly consider a continuum of firms each producing only a value $q \in Q$ that can be considered their type. In this situation each firm wants to match optimally with a consumer on the other side of the market. This formulation is consistent with different situations, think of a firm that wants to produce good and needs workers of an specific type to maximize their profits, or instead of firms and consumers, firms on both sides of the market that want to integrate in order to produce. The previous situations can be represented with the same model.

Now consider the utility possibility frontier for each side in our current problem:

$$\phi(q,\theta,u) = t(\theta) - cq \quad \text{s.t:} \quad \theta V(q) - t(\theta) = u$$

$$\Rightarrow \phi(q,\theta,v) = \theta V(q) - cq - u \tag{2.1}$$

$$\psi(\theta,q,v) = \theta V(q) - t(\theta) \quad \text{s.t:} \quad t(\theta) - cq = v$$

$$\Rightarrow \psi(q,\theta,v) = \theta V(q) - cq - v \tag{2.2}$$

From (2.1) and (2.2):

$$\phi(q,\theta,u) = \begin{cases} \underbrace{\theta V(q) - cq}_{f(q,\theta)} - u \text{ if } u \in [0, f(q,\theta)] \\ \\ \qquad 0 \qquad \text{if } u > f(q,\theta) \end{cases}$$

Here $\theta V(q)$ is the utility that a type θ gets in a match with a type q, c is the cost that a type q incurs when a matching is made and $t(\theta)$ is a non-linear tariff that side Q members of the market charges type θ.

Consider a matching rule $\alpha : \Theta \to Q$ and payoff functions $v : Q \to \mathbb{R}_+$ and $u : \Theta \to \mathbb{R}_+$. In equilibrium matches are formed optimally, meaning given $v(q)$ that for any $\theta \in \Theta$:

$$\theta \in \arg\max_{\tilde{\theta}} \left\{ \psi(\alpha(\tilde{\theta}), \theta, v(\alpha(\tilde{\theta}))) \right\} \tag{2.3}$$

from (2.3) we have:

$$\theta \in \arg\max_{\tilde{\theta}} \left\{ \theta V(\alpha(\tilde{\theta})) - c\alpha(\tilde{\theta}) - v(\alpha(\tilde{\theta})) \right\} \tag{2.4}$$

$$\Rightarrow \theta \in \arg\max_{\tilde{\theta}} \left\{ \theta V(\alpha(\tilde{\theta})) - t(\alpha(\tilde{\theta})) \right\} \tag{2.5}$$

If we consider $\alpha(\theta) \equiv q(\theta)$ then (2.5) reduces to (IC_θ) for each θ. This implies that the optimal matching can be achieved in this setting by a direct revealing mechanism.

We will restrict our attention only to differentiable mechanisms. Note that (IC_θ) constraint can also be written as:

$$\theta V(q(\theta)) - t(\theta) \geq \theta V(q(\tilde{\theta})) - t(\tilde{\theta}) \qquad \forall (\theta, \tilde{\theta}) \in \Theta \times \Theta$$

this implies that for every $\theta \in \Theta$ there is an infinity of constraints associated with (IC_θ). The following result solves that problem:

Proposition. 1. *A direct differentiable mechanism* $(q(\cdot), t(\cdot))$ *is incentive compatible if and only if the following conditions hold for all* $\theta \in \Theta$*:*

$$\theta V'(q(\theta)) q'(\theta) - t'(\theta) = 0 \tag{FOC}$$

$$q'(\theta) \geq 0 \tag{MON}$$

This result allow us to reduce an infinity of (IC_θ) constrains to just two (FOC) and (MON). Another important implication is that (MON) or monotonicity constraint guarantees that the optimal matching function $q(\theta)$ will be increasing, i.e. this is higher types will be matched with higher types, for example in the context of a labor market this means that the more productive firms will get the more productive workers.

In the next two section we will expose the problems that arise when we try to generalize this result to higher dimensions and our proposed modification to the original model. We will use this generalization to set a matching model with multi-dimensional types on each side of the market.

2.3 Multi-Dimensional Model

We will turn our attention to a multi-dimensional setting. Suppose now that types on both sides of the market are multi-dimensional. There are many situations in which one-dimensional types are not enough to capture the nature of the economic interaction. An example could be the situation proposed by (Lindenlaub, 2017) in which workers are endowed with cognitive and manual skills and firms are endowed with demand for cognitive and manual skills.

2.3.1 Problems that Arise when Increasing the Dimensionality

As we have seen before, multi-dimensionality can appear in both sides of the market, again, we can draw a parallel with the monopolist problem with a multi-dimensional product and type spaces. It is not straightforward to generalize results obtained in uni-dimensional models to the multidimensional case. Even the most simple notions of order and monotonic functions disappear when we are treating with more than one dimension.

Consider the case of a two-dimensional product (q, q') and a two-dimensional type (θ, θ'), it is clear that if $q << q'$ then q' is greater than q and therefore must produce higher utility but in the case that $q_1 < q_1'$ and $q_2 > q_2'$ then the notion of which consumption bundle is greater (therefore more desirable) depends on which characteristics measure each component. Recall our example of cars if q_1 measures the engine of the car and q_2 the stereo then which bundle is more desirable depends on how much the consumers value each of those characteristics.

Something similar happens in the type space, in the uni-dimensional case, types on both sides could be ordered, then we could make assertions like: "type θ is higher than θ'", in the multi-dimensional case this is no longer valid. Then we cannot longer guarantee that the "higher" type is paired with the higher "type", on the other side of the market.

We are interested in proposing a model that induces a natural ordering in both type spaces that allow us to state under what conditions we have *PAM*.

2.3.2 Proposed Multi-Dimensional Model

In this section we will propose a generalization to (Maskin and Riley, 1984), allowing both the product and the type space to be multi-dimensional Consider that our monopolist now produces a good with several characteristics: $q = (q_1, ..., q_m) \in R^n_+$ and faces consumers which differ on their type $\theta = (\theta_1, ..., \theta_n) \in \Theta \subset \mathbb{R}^n$. Now the utility of the principal is:

$$W(q,t) = t - c(q) \qquad \textit{with} \qquad \frac{\partial c}{\partial q_i} > 0 \quad \forall i = 1, ..., m$$

The function $c(\cdot)$ is the cost that the monopolist faces. For each agent will consider the following utility function:

$$U(q,t,\theta) = h(\theta)V(q) - t$$

Where $h : \mathbb{R}^n \to \mathbb{R}$ is a concave function that characterize how the type of an agent changes the utility she gets from consumption. $h(\cdot)$ is such that:

$$\frac{\partial h}{\partial \theta_i} > 0 \quad \forall i = 1, ..., n$$

we will also impose:

$$\frac{\partial V}{\partial q_j} > 0 \quad \forall j = 1, ..., m$$

Again we could draw a parallel with the matching setting, considering that firms on each side of the market value their interaction with a (multi-dimensional) type on the other side of the market, by protecting that type on to the real line. Such projections are through the functions $h(\cdot)$ and $c(\dot{})$. To illustrate this consider the utility possibility frontier for each side in the current problem:

$$\phi(q,\theta,u) = t(\theta) - c(q) \quad \text{s.t:} \quad h(\theta)V(q) - t(\theta) = u$$

$$\Rightarrow \phi(q,\theta,v) = h(\theta)V(q) - c(q) - u \tag{2.6}$$

$$\psi(\theta, q, v) = h(\theta)V(q) - t(\theta) \quad \text{s.t:} \quad t(\theta) - c(q) = v$$

$$\Rightarrow \psi(q, \theta, u) = h(\theta)V(q) - c(q) - v \tag{2.7}$$

From (2.6) and (2.7):

$$\phi(q, \theta, u) = \begin{cases} \underbrace{h(\theta)V(q) - c(q)}_{f(q,\theta)} - u \text{ if } u \in [0, f(q, \theta)] \\ \\ \quad 0 \qquad \text{if } u > f(q, \theta) \end{cases}$$

By slightly modifying the steps in (2.3)-(2.5) we can set the matching function $\alpha(\theta) \equiv q(\theta)$ then the optimal match reduces to (IC_θ) for this problem. This implies that the optimal matching can be achieved in this setting by a direct revealing mechanism. We are interested in direct mechanisms $g(\tilde{\theta}) = (q(\tilde{\theta}), t(\tilde{\theta}))$, where $\tilde{\theta}$ is the announced type of the agent.

Define the utility that the a type $\theta \in \Theta$ gets when anounces that her type is $\tilde{\theta}$:

$$U(\theta, \tilde{\theta}) = h(\theta)V(q(\tilde{\theta})) - t(\tilde{\theta})$$

Then an incentive compatibility restriction that any mechanism $g(\cdot)$ must satisfy is:

$$U(\theta, \theta) \geq U(\theta, \tilde{\theta}) \quad \forall \tilde{\theta} \in \Theta$$

this means that the higher utility that a type θ can get is when she reveals her true type, this can be reformulated as:

$$\theta \in \arg\max_{\tilde{\theta}} \{h(\theta)V(\tilde{\theta}) - t(\tilde{\theta})\} \tag{IC$_\theta$}$$

The following is a proposition that allows to replace the set of (IC_θ) restrictions with conditions on utility function.

Proposition. 2. *A differentiable mechanism $g(\theta) = (q(\theta), t(\theta))$ is incentive compatible if and only if the following holds:*

$$h(\theta)\nabla_\theta V(q(\theta)) - \nabla_\theta t(\theta) = 0 \in R^n \tag{2.8}$$

$$\nabla_q V(q(\theta))\nabla_{\theta_i} q(\theta) \geq 0 \quad \forall i = 1, ..., n \tag{2.9}$$

Where $\nabla_q V(q)$ and $\nabla_q t(q)$ is the gradient vector of $V(\cdot)$ and $t(\cdot)$ respect to θ and $\nabla_{\theta_i} q(\theta)$ is the $i - th$ row of the Jacobian Matrix of $q(\cdot)$.

Proof. We start by proving **sufficiency**, assume that (1) and (2) holds and there is some $\theta \in \Theta$ for which (IC_θ) does not hold. Consider w.l.o.g $(h(\theta) \leq h(\tilde{\theta}))$. Since $\tilde{\theta} \neq \theta$ there is a path between (a curve $\gamma(\cdot)$ with extreme points) $\gamma(0) = \theta$ and $\gamma(1) = \tilde{\theta}$, defined by:

$$\gamma : [0, 1] \rightarrow R^n$$

where:

$$\gamma(\lambda) = (1 - \lambda)\theta + \lambda\tilde{\theta}$$

$$h(\gamma(\lambda)) \geq (1 - \lambda)h(\theta) + \lambda h(\tilde{\theta}) \geq h(\theta)$$

$$h(\gamma(\lambda)) \geq h(\theta) \quad \forall \lambda \in [0, 1]$$

Define:

$$\varphi(x) = h(\theta)V(q(x)) - t(x)$$

since we are assuming that there is some $\theta \in \Theta$ for which (IC_θ) does not hold, then:

$$\exists \tilde{\theta} \in \Theta \textit{ such that } \varphi(\tilde{\theta}) > \varphi(\theta) \tag{2.10}$$

by the *The Fundamental Theorem of Line Integrals*:

$$\int_\gamma \nabla\varphi d\gamma = \varphi(\gamma(a)) - \varphi(\gamma(b)) = \varphi(\tilde{\theta}) - \varphi(\theta) > 0$$

with:

$$\int_\gamma \nabla\varphi d\gamma = \int_\gamma \nabla\varphi(\gamma(x))\nabla\gamma(x)dx$$

and:

$$\nabla\varphi(\gamma(x)) = h(\theta)\nabla V(q(\gamma(x))) - \nabla t(\gamma(x))$$

then: Since for every element in the curve $\gamma(\cdot)$ we have that $h(\gamma(x)) \geq h(\theta)$ and we have that every element of the vector $\nabla_q V(q(\theta))\nabla_\theta q(\theta)$ is positive (this is guaranteed by (2)), then we must have that:

$$\begin{aligned}[h(\theta)\nabla V(q(\gamma(x))) - \nabla t(\gamma(x))]\,\nabla\gamma(x)&\\ \leq [h(\gamma(x))\nabla V(q(\gamma(x))) - \nabla t(\gamma(x))]\nabla\gamma(x) &\qquad \forall x \in [0,1] \end{aligned} \tag{2.11}$$

therefore:

$$\begin{aligned}\int_\gamma \nabla\varphi d\gamma = \int_\gamma [h(\theta)\nabla V(q(\gamma(x))) - \nabla t(\gamma(x))]\nabla\gamma(x)dx&\\ \leq \underbrace{\int_\gamma [h(\gamma(x))\nabla V(q(\gamma(x))) - \nabla t(\gamma(x))]\nabla\gamma(x)dx = 0}_{\text{because (2.11) holds for all } x}&\end{aligned}$$

which contradicts (3).

Now to prove **necessity**, assume that for all θ (IC_θ) holds, then if we take any two $\tilde{\theta} \neq \theta$ the following must hold:

$$h(\theta)V(q(\theta)) - t((\theta)) \geq h(\theta)V(q(\tilde{\theta})) - t((\tilde{\theta}))$$

$$h(\tilde{\theta})V(q(\tilde{\theta})) - t((\tilde{\theta})) \geq h(\tilde{\theta})V(q(\theta)) - t((\theta))$$

Summing the above we obtain:

$$[h(\tilde{\theta}) - h(\theta)][V(q(\tilde{\theta})) - V(q(\theta))] \geq 0$$

If $h(\tilde{\theta}) > h(\theta)$, then we have:

$$V(q(\tilde{\theta})) > V(q(\theta))$$

By convexity of $V(\cdot)$ we get:

$$V(q(\tilde{\theta})) > V(q(\theta)) \qquad \Rightarrow \qquad \nabla_q V(q(\theta))(q(\tilde{\theta}) - q(\theta)) \geq 0$$

we can write $\tilde{\theta} = \theta + \beta$, where $\beta \in \mathbb{R}^n$. then in virtue of $q(\cdot)$ being a differentiable function:

$$\nabla_q V(q(\theta))\nabla_\theta q(\theta)\beta \geq 0 \qquad \Rightarrow \qquad \sum_{i=1}^{n} \nabla_q V(q(\theta))\nabla_{\theta_i} q(\theta)\beta_i \geq 0$$

We can choose to $\beta = e_i$ since we had imposed $\partial h / \partial \theta_i > 0$ then $h(\theta) < h(\theta + e_i) = h(\tilde{\theta})$, therefore:

$$\nabla_q V(q(\theta))\nabla_{\theta_i} q(\theta) \geq 0 \quad \forall i = 1, ..., n$$

$\square$

2.3.3 Implications of Projecting the Type Space

In the uni-dimensional case we had the following utility function:

$$U(q, t, \theta) = \theta V(q) - t$$

then we said that a type θ' was higher than a that a type θ if $\theta' > \theta$, which implies that: $U(q, t, \theta') > U(q, t, \theta)$.

In a multi-dimensional setting we cannot order the types but we can still compare utilities. We can then define an ordering that is consistent with the uni-dimensional case as:

$$\theta \preceq \theta' \Leftrightarrow U(q, t, \theta) \leq U(q, t, \theta')$$

which reduces to:

$$\theta \preceq \theta' \Leftrightarrow h(\theta) \leq h(\theta')$$

Based on this ordering we can partition the the type space:

$$\Theta_h = \{\theta \in \Theta | h(\theta) = \theta\}$$

This equivalence classes are consistent with *indifference sets* defined by (Chiappori et al., 2016). A nice property that arises from this order is that in equilibrium types belonging to the same equivalence class will receive the same allocation from the monopolist, this is summarized in the following proposition:

Proposition. 3. *Let $\theta_1, \theta_2 \in \Theta_h$ for some h, then an allocation $(q(\theta), t(\theta))$ is (IR_{θ_1}) and (IC_{θ_1}) if and only if it is (IR_{θ_2}) and (IC_{θ_2}) furthermore, in equilibrium, the monopolist can offer the same allocation to both types.*

Proof. The first part is trivial since for any two types $\theta_1, \theta_2 \in \Theta_h$ they get the same utility for any allocation.

Now consider that there are optimal allocations $(q(\theta_1), t(\theta_1))$ and $(q(\theta_2), t(\theta_2))$ for each type, we will take them as different:

$$q(\theta_1) \neq q(\theta_2) \text{ and } t(\theta_1) \neq t(\theta_2) \tag{2.12}$$

note that by the incentive compatibility constraint for each agent:

$$hV(q(\theta_1)) - t(\theta_1) \geq hV(q(\theta_2)) - t(\theta_2)$$

$$hV(q(\theta_2)) - t(\theta_2) \geq hV(q(\theta_1)) - t(\theta_1)$$

$$\Rightarrow h(V(q(\theta_1)) - V(q(\theta_2))) = t(\theta_1) - t(\theta_2)$$

which means that (2.12) must hold for the allocations to be different. Consider without loss of generality that:

$$t(\theta_1) - c(q(\theta_1)) > t(\theta_2) - c(q(\theta_2))$$

then the monopolist can set $(q^*(\theta_2), t^*(\theta_2)) = (q(\theta_1), t(\theta_1))$ which the agent would still accept and increase his profits, thus contradicting that the the allocation $(q(\theta_2), t(\theta_2))$ is optimal. $\square$

The implication of **Proposition. 3.** is that there is a function:

$$\alpha : \Theta_H \to Q$$

where $\Theta_H = \{\Theta_h | h \in \mathbb{R}_+\}$, that establish the matching rule between element of each equivalence class Θ_h and each type on Q.

Note that for every $\theta \in \Theta$ we can define $\alpha(\theta) \equiv \alpha(h(\theta))$, so we will consider $\alpha(\cdot)$ to be both a function from the equivalence classes set or from the type space.

2.3.4 Equilibrium

An equilibrium for this economy is a triplet (α, u, v) were α is the matching rule (generally a correspondence between type spaces), and the functions $u : Q \to \mathbb{R}$ and $v : Q \to \mathbb{R}$ are payoff functions. In our model the payoff function are determined by:

$$u(q) = t(\alpha^{-1}(q)) - c(q)$$

$$v(\theta) = h(\theta)V(\alpha(\theta)) - t(\theta)$$

Following (Legros and Newman, 2007) we will consider *stability* as the desired property of equilibrium in our analysis.

Definition. 1. *A match $\alpha : \Theta \to Q$ is said to be stable given payoff functions (u, v) if:*

$$\nexists (q, \theta) \in Q \times \Theta \text{ and } v > v(\theta) \text{ such that } \phi(q, \theta, v) > u(q) \tag{2.13}$$

The next proposition proves that any matching rule that is incentive compatible is stable by the definition in (2.13):

Proposition. 4. *Let $\alpha : \Theta_H \to Q$ a matching rule that meets (IC_θ), and payoff functions (u, v) specified in the model, then $\alpha(\cdot)$ is stable.*

Proof. Suppose there are $q \in$, $\theta \in \Theta$ and $v > v(\theta)$ such that $\phi(q, \theta, v) > u(q)$, then:

$$v > v(\theta) \quad \Rightarrow \quad v > h(\theta)V(\alpha(\theta)) - t(\theta) \tag{2.14}$$

$$\phi(q, \theta, v) > u(q) \quad \Rightarrow \quad h(\theta)V(q) - v > t(\alpha^{-1}(q)) \tag{2.15}$$

From (2.15) we have:

$$\phi(q, \theta, v) > u(q) \quad \Rightarrow \quad h(\theta)V(q) - t(\alpha^{-1}(q)) > v$$

and by (2.14):

$$h(\theta)V(q) - t(\alpha^{-1}(q)) > h(\theta)V(\alpha(\theta)) - t(\theta)$$

Which is a contradiction by (IC_θ). $\square$

Now we turn our attention to the monotony of the matching in equilibrium. As stated earlier, a monotonic matching is one that preserves ordering on the type-spaces. There is an clear difficulty when type-spaces are multi-dimensional, there is no complete ordering of multi-dimensional sets. We will define an ordering that is consistent with our theory and study the monotonic properties of the matching function under such order.

We will use the ordering of the type-space Θ provided by the projection function $h : \Theta \to \mathbb{R}_+$, for this order to make sense, it has to be that in equilibrium $h(\cdot)$ defines a preference relationship of the elements on the type-pace Q over the elements in type-space Θ. Then the preference relation:

$$\theta \preceq \theta' \quad \Leftrightarrow \quad h(\theta) \leq h(\theta')$$

has to be read as:

For each pair θ and θ', any type q will prefer to match with θ' if and only if $h(\theta) \leq h(\theta')$

For this to hold for every q can charge a higher tariff to elements of a higher equivalence class.

Proposition. 5. *In equilibrium:*

$$h(\theta) = h \leq h' = h(\theta') \quad \Leftrightarrow \quad t(h) \leq t(h')$$

Proof. We begin by proving necessity, consider the utility of an agent "lowest" class Θ_h, since for every agent the Individual Rationality constrain must hold, then:

$$U(h) = hV(q(h)) - t(h) \geq 0$$

now in virtue of the Incentive Compatibility constrain we have for all $h > h$:

$$U(h) = hV(q(h)) - t(h) \geq hV(q(h)) - t(h) > hV(q(h)) - t(h) \geq 0$$

Therefore it is only needed that (IR_h) holds, for all other (IR_h) to hold, furthermore in equilibrium the utility of the type h must be equal to zero. Also by (IC_h) we have:

$$U(h) = hV(q(h)) - t(h) = \max_{\tilde{h}}\{hV(q(\tilde{h})) - t(\tilde{h})\}$$

Then by the envelope theorem:

$$U'(h) = V(q(h))$$

which implies that:

$$U(h) = U(h) - U(h) = \int_h^h V(q(x))dx \equiv R(h)$$

We have follow the standard procedure to define the *Informational Rent* of elements of the class Θ_h, this implies that the informational rent does no varies within equivalence classes.

Now, to prove the necessity consider $h < h'$ and $q(h) = q(h')$ denote $V(q(h)) = V(q(h')) = V$ we can write each $t(\cdot)$ in terms of the informational rent:

$$t(h') = h'V - R(h')$$

$$t(h) = hV - R(h)$$

subtracting the above we have:

$$t(h') - t(h) = V(h' - h) - (R(h') - R(h)) = V(h' - h) - \int_h^{h'} V(q(x))dx$$

since $V(q(h)) = V(q(h'))$ then $R(h') - R(h)$ is the integral of an holomorphic function over a rectifiable path then by Cauchy's integral theorem:

$$R(h') - R(h) = \int_h^{h'} V(q(x))dx = 0$$

therefore:

$$t(h') - t(h) = V(h' - h) > 0 \qquad \Rightarrow \qquad t(h') > t(h)$$

To prove sufficiency suppose that in equilibrium there is are allocations $(q', t(h'))$ and $(q, t(h))$ with $t(h') \geq t(h)$ and $h' < h$ then by recall that when proving **Proposition. 2.** we arrive to an expression similar to this one:

$$[h - h'] [V(q) - V(q')]] \geq 0$$

and this hold whenever (IC_θ) holds, for all h, h' and q, q', now taking $t(h') > t(h)$ we have:

$$[h - h'] [V(q) - V(q')]] \geq t(h') > t(h)$$

by expanding the above we arrive at:

$$\underbrace{(h'V(q') - t(h'))}_{R(h')} - \underbrace{(hV(q) - t(h))}_{R(')} \geq \underbrace{h'V(q') + hV(q')}_{\geq 0}$$

Which implies that $R(h') \geq R(h)$ thus $h' \geq h$ since $R(h)$ is increases with h.

$\square$

Then it is the order proposed, represents a preference relationship over the set Θ. It is much clear that the order defined by the function $V : Q \to \mathbb{R}$:

$$q \preceq q' \quad \Leftrightarrow \quad V(q) \leq V(q')$$

represents preferences of elements of Θ over the type-space Q. Then endowing both type-spaces with this ordering we can formulate the following proposition regarding monotonicity of the matching:

Proposition. 6. *Let* $\alpha : \Theta_H \to Q$ *a matching rule that meets* (IC_θ)*, then* $\alpha(\cdot)$ *is* PAM.

Proof. The proof if this proposition is trivial since (IC_θ) implies that:

$$[h(\theta') - h(\theta)][V(\alpha(\theta')) - V(\alpha(\theta))] \geq 0$$

which means that for every $\theta \preceq \theta'$ then $\alpha(\theta) \preceq \alpha(\theta')$.

$\square$

Chapter 3

Application: Signaling Game

This chapter presents an application of our results to a signaling game with multi-dimensional types. We will propose a model based on (Mailath, 1987).

Consider the interaction between an agent informed about their type (sender) and an uninformed one (receiver). The receiver can infer the type of the sender through an action. Let $U(\theta, \tilde{\theta}, y)$ be the the utility of the sender of type θ when the receiver infer that her type is $\tilde{\theta}$, and she undertake an action y we will assume that the utility function has the form:

$$U(\theta, \tilde{\theta}, y) = h(\theta)V(q(\tilde{\theta})) - t(y)$$

The set of possible types are $\Theta \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ is the set of all possible actions that the receiver can undertake based on the inferred type of the sender. Consider an strategy $\tau : \Theta \to \mathbb{R}$ a strategy for the sender then the receiver infers her type as $\tau^{-1}(y)$. (Mailath, 1987) states that the strategy is a *separating equilibrium* if it is one-to-one and satisfy incentive compatibility according to:

$$\tau(\alpha) \in \arg\max_{y \in \tau(\Theta)} \left\{ U(\theta, \tau^{-1}(y), y) \right\} \tag{3.1}$$

Clearly the restriction of τ being one-to-one rules out the possibility of obtaining a separating equilibrium over the set of types Θ. But in the previous chapter we prove that under the specifications of our model it is enough to separate the space in equivalence classes. Even if our results don't allow us to find a separating equilibrium on Θ, we can find it on Θ_H.

We can define $\tau(\theta) \equiv h(\theta)$, meaning that instead of sending a signal about their type agents send a signal about their equivalence class. Then by **Proposition. 3.** they will receive the same utility in equilibrium.

Chapter 4

Non-Transferable Utilities NTU

The model we proposed is useful to describe a situation where partners can share the match output using transfers. This is due to the fact that utility functions are linear in money (Chade et al., 2017). OIn other words that the TU matching situation can be identified with a screening model where both the principal and the agent's utility functions are quasi-lineal in $t \in \mathbb{R}_+$. Even if such a model useful in many situations, it can only explain so much, there are many practical applications where the assumption of transferable utilities fail.

We will propose a more general model of screening when utility functions are not quasi-linear in money. The particularity of this model is that it maintains projections of type-space using a function $h : \Theta \to \mathbb{R}$. The implementability conditions for allocations of the screening model which are presented as in (Basov, 2006). As exposed by (Chiappori et al., 2016) closed solutions for multi-dimensional models are impossible to find. The following section outlines how we can present a similar model in this more general case.

4.1 Screening Model without Quasi-Lineal Functions

In this section we will follow (Basov, 2006) Chapter 8. The following is intended both as a road map on how we would like to model the NTU situation and to illustrate the intractability and difficulty of the more general case.

We present a modified version of a general multi-dimensional model, we maintain projections both of the type-space and product-space on to the real line in order to be consistent with previous chapters. Since we are considering a screening model of a monopolist we will keep the assumption of quasi-linearity in one side of the market in this scope this means that the principal's utility function will remain the same. On the other side of the market we are droping linearity in t i.e. the agent's utility function is:

$$U(q, \theta, t) = h(\theta)V(q, t)$$

where $u(\cdot, \cdot, \cdot)$ is twice continuously differentiable in q and θ, strictly increasing in $V(\cdot)$ and $h(\cdot)$ and strictly decreasing in t. We will also assume that for any continuous function $s(\theta)$, there is a function $\tau(q, \theta, s(\theta))$ that is the only solution to:

$$s(\theta) = h(\theta)V(q, \tau(q, h(\theta), s(\theta))) \tag{4.1}$$

$s(\theta)$ is the surplus function of a type θ *i.e.* the maximum utility that a particular type can obtain. We are interested in the class of implementable surplus functions:

Definition. 2. (Basov, 2006), p. 182 *A surplus function* $s(\cdot)$ *is called implementable if there exists a measurable tariff* $t(\cdot)$ *such that:*

$$s(\theta) = \max_{q \in Q} h(\theta)V(q, t(q))$$

(Basov, 2006)also provides a result that allow us to characterize every implementable surplus function:

Theorem. 1. (Basov, 2006), p. 182 *A surplus function $s(\theta)$ is implementable if and only if:*

$$s(\theta) = \sup_{q \in Q} \left\{ h(\theta) V \left(q, \max_{\tilde{\theta} \in \Theta} \tau(q, \tilde{\theta}, s(\tilde{\theta})) \right) \right\}$$

In the context of screening we are also interested with *allocations*, which in the matching setting will be the matching function between each side of the market, like in the case of surplus functions we are interested on implementable allocations:

Definition. 3. (Basov, 2006), p. 183 *An allocation $q(\theta)$ is called implementable if there exists a measurable tariff $t(\cdot)$ such that:*

$$q(\theta) \in \arg\max_{q \in Q} h(\theta) V(q, t(q))$$

for any $\theta \in \Theta$

Let $q(\cdot)$ be a implementable allocation, implemented by $t(\cdot)$ and $s(\cdot)$ the surplus, then applying the generalized envelope theorem by (Milgrom and Segal, 2002) we can obtain:

$$\nabla s(\theta) = \left(\frac{\partial}{\partial \theta_i} (h(\theta) V(q(\theta), \tau(q, h(\theta), s(\theta))) \right)_{i=1}^{n} \tag{4.2}$$

where $\tau(\cdot, \cdot, \cdot)$ is defined by (4.1). This system of partial differential equations is key in order to characterize the implementability of the allocations.

Theorem. 2. (Basov, 2006), p. 186 *For an allocation $q(\cdot)$ to be implementable the system* (4.2) *must be compatible (i.e. have a unique solution). If* (4.2) *is compatible then $q(\cdot)$ is implementable if and only if it is implementable by the tariff:*

$$s^*(q) = \max_{\theta \in \Theta} \tau(q, \theta, s(\theta))$$

where $s(\theta)$ is a solution to (4.2).

Chapter 5

Concluding Remarks

There are many situations where representing types with a single index is an oversimplification. As we have shown some matching models are equivalent to screening models, in these cases, we can characterize possible matching as being Incentive Compatible. In this work, we propose a multi-dimensional screening model. Our strategy to deal with multi-dimensionality is to project into the real line. This strategy reduces the scope of the model, but give us enough tractability. Under those assumptions, we proved that matching is stable if and only if they are incentive compatible. The use of projections allowed us to endow type-spaces with an order. Those order relations were consistent with our theory and served as a tool to reinterpret the notion of monotonicity; in particular, we prove that any incentive compatible matching is PAM.

As shown before implementation challenges of the NTU model, can hinder their applicability to real life situations. To overcome this an idea is to consider a characteristic pair as numerary and express all other pairs valuations as relative to that pair. This in principle, can allow us to convert a NTU model in to an TU one. and is in our opinion an interesting direction for future work.

The other way to deal with the problem of NTU models is to solve, what we consider to be the main shortcoming of this work: the lack if a theoretical a result for the NTU case. As shown in the preceding chapter, the implementability conditions of multi-dimensional screening models when utility functions are not quasi-linear are convoluted. As difficult as they are, this kind of models present a viable path to study the properties of matching models with NTU. In order to make use of the multi-dimensional screening model to study NTU matching, we will need two results. The first one is that under the conditions stated by (Basov, 2006) there is an implementable allocation between equivalences classes defined by projection functions. Secondly, we need to prove that this allocation is stable and monotonic. Intuition in this regard points to finding a relation between single-crossing like conditions on the utility functions in order to prove PAM. One possible path is to prove that Generalized Increasing Difference condition proposed by (Legros and Newman, 2007) holds when we have a matching between equivalence classes. Those remain challenging problems for later investigation.

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