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CONTRACTING IN TEAMS WITH NETWORK TECHNOLOGIES

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PRESENTA

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A mi par intelectual,
por convertir cada día en una aventura.

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Abstract

We develop a contracting model between the owner and the workers of a firm when production depends directly on a network of synergies among workers. We aim to answer how the owner of the firm uses the network structure to maximize profits. With this purpose, we analyze two contracting regimes: single wage and perfect discrimination. We find that individual network characteristics, as well as aggregate measures, affect profits and salaries. We also study the parameters for which the incentives to discriminate and to account for the network structure are significant.

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Chapter 1

Introduction

Since working in teams is one of the most common forms of organizing tasks in organizations, the question of what makes them operate effectively matters. That is why companies like Google have spent significant resources in finding out the answer to that question. Projects Oxygen and Aristotle carried out in Google have analyzed leadership and team members, respectively, to discover the secrets of team effectiveness. One answer that emerged was that group norms define how teams work, see Duhigg (2016). Incentive design is one important group norm, but an unavoidable consequence of joint production is that profits must be shared among the parties involved. Furthermore, when team members' effort cannot be contracted upon, problems such as free-riding generate severe efficiency loss (Hölmstrom (1982)).

The heterogeneous interactions among team members and the structure of the team suggest that understanding the complexity of the network embedded in the team is crucial to devise group norms that help improve the team's effectiveness. In this thesis, we analyze a situation where a firm maximizes profits when the underlying social structure produces synergies in team members' efforts. We pay special attention to cases where the impact of effort vary across individuals, depending on their degree of group exposure (i.e., the number of connections of each worker) and the intensity of their interactions. This positive effect is referred to in the literature as local network externalities. Mainly, synergies in the workplace are modeled by having actions of individuals affecting the payoffs of their peers directly (Segal, 1999; Bernstein and Winter, 2012; Belhaj and Deroïan, 2019). In contrast, we focus on situations in which the synergies affect the outcome of production directly, and the payoff of workers indirectly.

The objective of this thesis is to answer the following questions. In a joint enterprise with complementarities in production, what is the best contract scheme to maximize profits? Moreover, how does the owner of a firm incorporate the network technology to boost profits? Finally, under what circumstances is taking into account the network structure imperative? To address these points, we analyze a principal-multiagent problem of contracting when there are synergies among team members. In our framework, these complementarities arise from collaboration and teamwork. Consequently, we model synergies as part of the production function. The owner of the firm designs a set of incentive contracts to ensure team members exert the optimal efforts, that is, the efforts that maximize profits. Nonobservable efforts, and the analysis of two contract schemes: (i) single uniform wage, and (ii) perfect discrimination of salaries, characterize our setting. In the single-wage scheme, the firm's owner is constrained to pay the same salary to each worker. In the wage discrimination case, the principal is able to pay different salaries to each team member. By means of a numerical exercise, we compare the results of these two schemes and discuss the situations in which knowing the network structure is most valuable.

We find that under the single wage regime, salaries and efforts are homogeneous, and the owner of the firm employs aggregate measures of the network structure to induce optimal efforts. When discrimination of wages is possible, the firm's owner uses information on the centrality of workers to choose efforts and wages. We also compare the two regimes, and characterize the parameters for which no discrimination yields the highest profits. It is intuitively clear that the disparities between these two cases are due to the ability to use the network structure to the full extent, which is only possible when the owner discriminate wages.

In general, we find that incentives to discriminate and use network information are more significant when the synergies are large. Nonetheless, in the space of parameters we analyze, we discover that the profit gap between the two regimes is notably small. We believe that this result has the following consequences. The incentives to discriminate are smaller when (i) there are no significant variations among workers' centralities (ii) the cost of researching the particular structure of the network is high, and (iii) there are ethical reasons against discrimination. If the gains in productivity derived from discrimination are not significant, the firm is better off not discriminating wages in the account of network position. We relate profits to aggregate measures of the network as a guide to when to discriminate salaries. Therefore, this allows us to assess the profitability of bearing the informational costs of researching a particular network technology. Since we model situations in which the nature of the productive activities is impacted directly by the network, we contribute by broadening the applications and demarcation of network technologies.

The rest of the thesis is organized as follows. Chapter 2 provides a review of the related literature. In Chapter 3, we introduce the general model, with the two restrictions, single wage and no discrimination. In Chapter 4, through examples and simulations, we examine the differences between the contracting regimes and discuss the relevance of accounting for the network structure. Finally, Chapter 5 concludes. Proofs appear in the Appendix.

Chapter 2

Related Literature

The current work is related to three branches of literature: contracting in principal-agent models, network externalities, and network games.

This thesis is part of an extensive body of literature on optimal contracting in principal-agent models. This literature goes back to the 1970s. Prominently, Hölmstrom (1979) analyzes the role of imperfect information when a principal contracts with a risk-averse agent subject to moral hazard. He establishes that when the action is not observable, optimal contracts will be second-best due to the trade-off between risk-sharing and incentives. Many significant works have studied contracts in the context of teams in the past decades. Some of these include Hölmstrom (1982), Rasmusen (1987) and Rayo (2007). Following Hölmstrom (1982), we mean by teams, a group of agents who organize collectively to engage in productive activities.[1]. This influential paper on moral hazard in teams proposes a budget breaker (principal) who punish the agents when profits are low, solving, this way, the incentive problem.

This work is in line with the literature on contracting with externalities. An environment where the principal commits to publicly and privately observed bilateral contracts on a multi-agent framework is studied by Segal (1999). In the first case, inefficiencies occur due to the externalities on agents' outside options, and in the second case, inefficiencies are due to the externalities at efficient outcomes. Furthermore, Segal (2003) considers a variety of situations

[1]We do not assume that agents share a common goal as in Marschak and Radner (1972)

when the principal can and cannot coordinate agents or discriminate contracts with both negative and positive network effects. He shows that in the setup where the principal is unable to coordinate individuals to choose his most preferred equilibrium, the principal benefits from using a divide-and-conquer mechanism. These models are developed under complete information and in an all-to-all or complete graphs. Heterogeneous and asymmetric externalities are studied in Bernstein and Winter (2012). By following Segal (2003), these authors focus on circumstances where the principal cannot coordinate agents to his preferred equilibrium. In this thesis, externalities in contracting and coordination among agents are analyzed under a more general network structure, which serves to undercover the impact of various network settings on the optimal contract.

Bernstein and Winter (2012) propose a model that considers heterogeneous and asymmetric externalities. Following Segal (2003), the authors focus on situations where the principal cannot coordinate agents to his preferred equilibrium. They solve for any matrix of externalities but in a more restrictive environment (binary effort).

The present work is also related to the branch of literature on network effects. Strategic complementarity, which is the inherited characteristic of positive network effects, was studied by Katz and Shapiro (1985) and Farrell and Saloner (1985). Farrell and Saloner (1985) study a game of incomplete information with positive adoption externalities in which firms determine whether to change to a new standard. They prove that there is a unique symmetric Bayes-Nash equilibrium that depends on the firm's type. Katz and Shapiro (1985) develop a static model of oligopoly to analyze markets in which consumption externalities are present. In these abovementioned papers, as in most of the existing literature[2], the network effects correspond to complete networks. In contrast, in the current setting, agents interact locally according to the network of synergies.

In the growing body of literature of network games with strategic interactions (Ballester et al., 2006; Corbo et al., 2007; Galeotti et al., 2010), payoff interdependence is usually driven by

[2]For more literature where network effects are studied in all-to-all networks, see Dybvig and Spatt (1983); Katz and Shapiro (1986)

the network structure of players' links[3]. In our model, the network does not affect the utilities of the agents in the usual fashion; nevertheless, it has an indirect effect on the individual's payoffs. Ballester et al. (2006) model a non-cooperative network game with local complementary payoffs where players exert an action simultaneously. As is common in this branch of the literature, the authors use linear-quadratic utilities. They find that the Nash equilibrium action of each player is proportional to his Bonacich centrality in the network of local complementarities. The Bonacich-Nash relation means that aggregate equilibrium increases with network size and density. They go further an investigate a policy that consists of targeting the player who, once excluded, leads to the optimal change in aggregate activity, i.e., the key player. To characterize the network optimal targets, they propose a new measure of network centrality, the inter-centrality measure. The concept of network centrality is crucial for designing discriminating contracts in the context of the model developed in this thesis.

A setup where the firm (principal) does not have perfect knowledge of the social structure but knows the degree distribution of customers (agents) is studied by several papers, such as Sundararajan (2007), Galeotti et al. (2010), Fainmesser and Galeotti (2016). These articles also explore situations where the agents have incomplete information about the interaction of others; that is, incomplete information of network links. These settings could provide a more realistic analysis of the relationship between consumers and firms. However, the firm-workers relation that the present work intends to examine is not accurately captured by such settings.

The amplification of an agent's actions is captured by the lowest eigenvalue of the matrix associated with the network (Bramoullé et al. (2014)). In this framework, the results of Ballester et al. (2006) concerning Katz-Bonacich Centrality do not hold. These results could be applied to RD spillovers and crime. Other authors, such as Candogan et al. (2012) and Bloch and Quérou (2013), study the problem of optimal monopoly pricing when consumers are embedded in a social network. These papers also find in slightly different setups, that the consumers' Katz-Bonacich centrality is related to optimal equilibrium. These papers differ from this thesis in the

[3]See Jackson and Zenou (2015) for an overview of games where a network structure connects players.

motivation, model, and structure of the game: they propose a monopoly-consumers relationship with a network good.

The works of Jadbabaie and Kakhbod (2019) and Belhaj and Deroïan (2019) are recent papers that situate themselves in the intersection of the aforementioned branches; in consequence, they are closer to our work. Jadbabaie and Kakhbod (2019) considers optimal contracting between a firm selling a good with positive network effects and consumers in a social network. Their contribution to the existing literature is that the valuation of each agent about network externality is uncertain to the firm and other agents. This article, likewise to Candogan et al. (2012) and the present work, emphasizes the relationship between profits and aggregate characteristics of the firm, such as the largest and smallest eigenvalues. Belhaj and Deroïan (2019) is concerned with group targeting in the presence of synergies characterized by a network. They focus on the impact of synergies between contracting and noncontracting agents when reservation utilities are endogenous. They show that under these assumptions, the principal can abstain from contracting with central agents. In our model, central agents bring the highest share of profits and are essential for the production process.

Chapter 3

Model

Consider an owner of a firm (principal) that maximizes profits and contracts with $N = \{1, ..., n\}$ workers (agents). There is an underlying social network shaping the way synergies between workers influence production. The adjacency matrix G encapsulates the network of synergies where $g_{ij} \in \mathbb{R}_+$ represents the intensity in which j's effort enhance i's effort in the outcome of production. We adopt the convention of $g_{ii} = 0$.

Worker i exerts nonobservable effort $a_i \in \mathbb{R}_+$ where A is a compact subset. The agents' effort vector is $a = (a_1, ..., a_n)$.

When efforts are nonobservable, the set of contracts offered by the owner of the firm is the set of functions $\{w_i(y)\}_{i \in N}$ in which agent i receives a compensation scheme $w_i(y(a))$ that depends on production y. The owner of the firm cannot punish the workers (limited liability), i. e. , $w_i \geq 0$. Also, $\bar{u_i}$ is agent i's outside option. The utility of worker i depends on his effort and the compensation that he receives from the owner of the firm, so that

$$u_i(a_i, w_i) = w_i - \psi_i(a_i)$$

where $\psi(a_i)$ is the disutility of effort of agent i which is convex and strictly increasing in a_i, and w_i is the compensation given by the owner of the firm.

The owner of the firm has perfect knowledge of the network and knows that the network of synergies g affects the outcome of production. One important feature of our model is that production function $y_g(a)$ depends directly on the network structure, as follows

$$y_g(a) = f\Big(\sum_i a_i\Big) + \gamma \sum a_i h_i\Big(\sum_j g_{ij} a_j\Big) \tag{3.1}$$

where $\gamma \geq 0$ is the strength of network effects, f and h_i are strictly non-decreasing in their inputs for all $i \in N$. The outcome of production is additively separable in the sum of efforts and the synergies resulting from the combined work of each agent and the rest.

Hence the utility of the profit maximizer principal is

$$V(a, w) = y_g(a) - \sum_i w_i \tag{3.2}$$

and she is subject to a budget constraint

$$\sum_i w_i(y_g) \leq y_g \tag{3.3}$$

Because we focus on the network effects, we assume all agents are homogeneous in other respects. Therefore, throughout this thesis, we assume that workers are identical, except for their position in the production structure. For now on, $\psi(a_i) = \beta a_i^2$ where β is a parameter of cost of effort for agents, and $\bar{u}_i = 0$ for all i. The production will be given by

$$y_g(a) = \sum_i a_i + \gamma \sum_i \sum_j g_{ij} a_i a_j$$

In the preceding equality, the functions f and h of the general equation3.1 are linear in their respective arguments.

3.1 Efficient Solution

If no moral hazard problem ware present, the principal could use an enforcing contract to guarantee that the optimal effort is exerted by the worker(see Hölmstrom (1979)). Then, the owner of the firm will offer agent i a compensation scheme $\{(w_i, a_i), (0, 0)\}$, where $a_i \in A$ specifies the effort demanded to the agent and $w_i \in \mathbb{R}_+$ the payment.

3.1.1 Single Wage

In this subsection, we turn our attention to a situation where the owner of the firm can only pay a single wage to all agents. In this benchmark model, for any effort and wage, the utility function of worker i is

$$u(a_i, w) = w - \beta a_i^2 \tag{3.4}$$

and the owner of the firm solve

$$\max_{w \geq 0} \; y_g(a) - \sum_i w_i$$

$$u_i(a_i, w) \geq 0, \qquad i \in N \tag{IR_i}$$

where (IR_i) is the individual rationality constraint of agent i.

To ensure that positive efforts are a feasible solution, we make the following assumption.

Assumption 1. ***(Bounded effort)*** *For all i, $\gamma \sum_i g_{ij} < \beta$ and $\gamma \sum_i g_{ji} < \beta$*

First, note that (IR_i) binds for all i. It is trivial that it must bind for the worker from whom the owner of the firm demands the highest effort a_j. Now, take a worker j for which $w > \beta a_j$, the owner of the firm can increase his utility by asking j to exert more effort until (IR_j) binds. This equality also implies that all agents will exert the same effort a. By substituting (IR_i) in the principal's payoffs, the program simplifies to choosing a such that it maximizes

$$
\begin{aligned}
& y_g(a) - n\beta a^2 \\
= na + a^2 & \Big[\gamma \sum_i \sum_j g_{ij} - n\beta\Big]
\end{aligned} \tag{3.5}
$$

Hence, the optimal condition of effort is given by

$$
n + 2a\Big[\gamma \sum_i \sum_j g_{ij} - n\beta\Big] = 0 \tag{3.6}
$$

After some manipulations, we can confirm the results of Proposition 1.

Proposition 1. *The noncooperative game with observable efforts and a single wage has an equilibrium where all workers accept and the optimal efforts and wages are given by*

$$
a^* = a_i^* = \frac{1}{2(\beta - \frac{\gamma}{n}\bar{g})} \qquad \forall i
$$

and

$$
w^* = \frac{\beta}{4(\beta - \frac{\gamma}{n}\bar{g})^2}
$$

where $\bar{g} = \sum_i \sum_j g_{ij}$.

To facilitate the analysis of the relationship between the optimal profits and the parameters, we focus only on β, γ, and n. That is because we could normalize the sum of weights ($\bar{g}$) to 1, by substituting γ for $\bar{\gamma} = \gamma\bar{g}$. In consequence, the optimal strategies are only positively affected by the intensity of interactions($\bar{\gamma}$), not by the structure of the network. Intuitively, the owner of the firm is not able to make use of the particular structure of the work interactions, when paying the same wage to all workers and demanding the same effort. So, the agent's optimal effort depends on the cost of efforts, the intensity of network effects, and the number of workers.

Moreover, the optimal efforts are monotonically decreasing in the disutility of effort and the number of workers, and monotonically increasing in the intensity of network effects. That is always true when assumption 1 holds. The same comparative statistics are valid for the optimal wage.

We characterize the firm's profits in the next proposition.

Proposition 2. *Under assumption 1, the optimal profits for the case of a single wage is*

$$\pi_s^* = \frac{n^2}{4(n\beta - \gamma\bar{g})}$$

To understand the relationship of the firm's optimal profits with each of these variables, we make use of some comparative statics results:

$$\frac{\partial \pi^*}{\partial \beta} = \frac{-n^3}{4(\gamma\bar{g} - \beta n)^2} < 0$$

$$\frac{\partial \pi^*}{\partial \gamma} = \frac{\bar{g}n^2}{4(\gamma\bar{g} - \beta n)^2} > 0$$

The partial derivative of optimal profit with respect to the cost of effort and strength of interactions is negative whenever the conditions for a bounded effort are met. That is, when assumption 1 holds.

$$\frac{\partial \pi^*}{\partial n} = \frac{n(\beta n - 2\bar{g}\gamma)}{4(\gamma\bar{g} - \beta n)^2}$$

On the other hand, the partial derivative of optimal profits with respect to the number of workers suggests that the relationship is positive when the inequality $n > \frac{2\bar{g}\gamma}{\beta}$ is met. In other words, ceteris paribus, increasing the number of workers, produces a decline in profits when n satisfies $\frac{\bar{g}\gamma}{\beta} < n < \frac{2\bar{g}\gamma}{\beta}$, and when $n > \frac{2\bar{g}\gamma}{\beta}0$, causes a decrease in profits. The previous reasoning is the equivalent of hiring new workers that do not integrate with the rest of the company collaboratively. This result indicates that increasing the number of workers without proportionally strong interactions between workers eventually leads to a fall in profits. This result implies that there is an optimal size of workers. Also, note that if we assume that the new workers are equally "integrated" to the company, the new addition to the company always yields higher profits.

The following Proposition includes comparative statistics of the profits in terms of the structure of the network.

Proposition 3. *Consider a single wage scheme with network of synergies represented by G. Suppose the weight of the interaction from j to i is increased in c such that $g'_{ij} = g_{ij} + c$ in a way that Assumption 1 is preserved, then $\pi'_s > \pi^*_s$.*

The proof of this result follows from Proposition 2, and it is omitted. Intuitively, to strengthen the links of the network produces a higher output. This boost salaries and demanded effort, which further raises profits.

The main takeaway of this section is that with a restrictive contract structure like the single wage setting, the owner of the firm is unable to take advantage of the position of the workers in the network of interactions. Moreover, any detail of the network structure other than the total sum of weights and the strength of network effects, it is not useful for the owner of the firm.[1] As a consequence, the profits do not depend on explicit knowledge of the particular network architecture; ergo, the owner of the firm, only employs aggregate network characteristics.

3.1.2 Wage Discrimination

We now explore situations in which the owner of the firm can discriminate among workers, i.e., the owner of the firm offers each worker i a wage w_i in exchange for an effort a_i, and the possibility of rejecting the job.

Before all else, we introduce some notation that will be used from now on. If A is a square matrix or a column vector, then A^T denotes its transpose. A bold letter indicates a column vector as in $\mathbf{a}$. The expression $A \circ B$ denotes the Hadamard product between the matrixes of the same dimension A and B; the entry i, j of $A \circ B$ is the element $a_{i,j}b_{i,j}$ where $a_{i,j} \in A$ and $b_{i,j} \in B$. For any vector $\mathbf{v}$, we let $D(v)$ denote the matrix with the elements of $\mathbf{v}$ in the diagonal and the entries outside the diagonal are all zero. Finally, $\mathbf{1}$ denotes the column vector of ones, $\mathbf{0}$ denotes the column vector of zeros, and $\mathbf{I}$ denotes the identity matrix.i

[1]Recall that we decided to separate the strength of network interaction from the weights. This is equivalent to work with the matrix $G' = \gamma G$.

In this section, we focus on interior solutions were efforts and salaries are positive. Namely, all agents accept, and the salaries are positive. We discuss the implications of some workers not being active in the company in the proof of Proposition 4, Appendix A. In summary, the relation between the network of synergies and optimal wages, remains for the subset of workers with positive efforts. For ease of exposition, we assume that the optimal set of workers of the company is N.

The owner of the firm solves a program similar to the single wage setting. Likewise, the participation constraints (IR_i) bind, which in this case does not imply that all efforts are equal,

$$\psi(a_i) = \beta a_i^2 = w_i \tag{3.7}$$

Substituting the previous equation in the objective function results in

$$\sum_i a_i + \gamma \sum_i g_{ij} a_i a_j - \sum \beta a_i^2 \tag{3.8}$$

The first order conditions derive in a system of linear equations:

$$1 + \gamma \sum_j (g_{ij} + g_{ji}) a_j - 2\beta a_i = 0 \qquad , \forall i \in N \tag{3.9}$$

To simplify the analysis and assure bounded variables, we suppose Assumption 1 as in the last section. The next lemma can be applied when Assumption 1 holds; the proof is in Appendix A.

Lemma 1. *If for all i, $\gamma \sum_j g_{ij} < \beta$ holds, then, the largest eigenvalue of matrix $G + G^T$, λ_{max}, satisfies $2\beta > \lambda_{max} * \gamma$*

We call the matrix $G + G^T$ *symmetrized network* of G. Since $G + G^T$ is symmetric, all the eigenvalues are real numbers, and, in particular, the maximum eigenvalue.

Expressing the system in matrix form, we get the following equation.

$$\mathbf{1} + \gamma(G + G^T)\mathbf{a} - 2\beta\mathbf{a} = 0 \tag{3.10}$$

In the following proposition, we characterize the optimal efforts and wages of the game.

Proposition 4. *Under Assumption 1, the noncooperative game with observable efforts and a wage discrimination has an equilibrium where all workers accept given by*

$$\mathbf{a}^* = [2\beta\mathbf{I} - \gamma(G + G^T)]^{-1}\mathbf{1}$$

$$\mathbf{w}^* = \beta[2\beta\mathbf{I} - \gamma(G + G^T)]^{-1}\mathbf{1} \circ [2\beta\mathbf{I} - \gamma(G + G^T)]^{-1}\mathbf{1}$$

To better illustrate the effects of the network, we provide the definition of the Katz-Bonacich Centrality, see Bonacich (1987). This popularity metric of the nodes of a network tells us about how well connected are the neighbors of a node. Armed with the concept of Katz-Bonacich, we present the optimal efforts and wages in Proposition 5.

Definition 1. *For a network with adjacency matrix G and scalar δ, the Katz-Bonacich centrality is a vector of parameter δ given by*

$$\mathcal{K}(G, \delta) = (I - \delta G)^{-1}\mathbf{1}$$

when $(I - \delta G)^{-1}$ is well define and nonnegative.

Proposition 5. *Under assumption 1., the noncooperative game with observable efforts and wage discrimination has an equilibrium where all workers accept and strategies are expressed in their Katz-Bonacich centralities:*

$$a^* = \frac{1}{2\beta}\mathcal{K}\Big(G + G^T, \frac{\gamma}{2\beta}\Big)$$

and

$$w^* = \frac{1}{4\beta}\mathcal{K}\Big(G + G^T, \frac{\gamma}{2\beta}\Big) \circ \mathcal{K}\Big(G + G^T, \frac{\gamma}{2\beta}\Big)$$

The previous result tell us that the efficient solution is to assign the actions of the agents proportional to their Katz-Bonacich centralities of the symmetrized network $G + G^T$. The Bonacich centrality of a worker i, $\mathcal{K}_i(G+G^T, \frac{\gamma}{2\beta})$ can be interpreted as the discounted(discount factor is $\frac{\gamma}{2\beta}$) sum of walks(weighted) that start with worker i in the network.[2] To better interpret this notion, it could be helpful to use the power series expansion the matrix of centralities $K(G+G^T, \frac{\gamma}{2\beta})$. That is, $K(G+G^T, \frac{\gamma}{2\beta}) = [I - \frac{\gamma}{2\beta}(G+G^T)]^{-1} = \sum_{k=0}^{\infty} \left(\frac{\gamma}{2\beta}\right)^k (G+G^T)^k$. Thus, the Katz-Bonacich centrality of individual i is $\mathcal{K}_i(G + G^T, \frac{\gamma}{2\beta}) = \sum_i \sum_{k=0}^{\infty} \left(\frac{\gamma}{2\beta}\right)^k (G + G^T)^k$. What the previous expression reveals, is that the Bonacich centrality measure is proportional to the expected number of visits discounted by $\left(\frac{\gamma}{2\beta}\right)^k$ at time k of a random walk in the symmetrized network. Intuitively, the symmetrized network gives us a measure of the number of work interactions two individuals have.

The insights from the case of a single wage concerning the parameters β, γ, and n continue to hold. That is because the centralities of each agent are non decreasing in the discount factor. We can verify this observation in the later analysis of the Katz-Bonacich centrality.

Next, we provide two equivalents characterization of the optimal profits for the case of wage discrimination, which we denote π_d.

Proposition 6. *Under Assumption 1, the optimal profits for the case of wage discrimination is*

$$\pi_d = \Big[\mathbf{1}^T + ([2\beta\mathbf{I} - \gamma(G + G^T)]^{-1}\mathbf{1})^T[\gamma G - \beta I]\Big][2\beta\mathbf{I} - \gamma(G + G^T)]^{-1}\mathbf{1}$$

which in terms of Katz-Bonacich centralities is

$$\pi_d = \Big[\mathbf{1}^T + \frac{1}{2\beta}\mathcal{K}\Big(G + G^T, \frac{\gamma}{2\beta}\Big)^T[\gamma G - \beta I]\Big]\frac{1}{2\beta}\mathcal{K}\Big(G + G^T, \frac{\gamma}{2\beta}\Big)$$

The proof follows from Proposition 4, and it is omitted. In subsequent sections, we will

[2] A walk between node x and y in a graph is a list of nodes where each node is connected with his successor, and starts with x and ends in y.

further discuss the dynamics of the profits for the discrimination case. The following corollary for the case of symmetric networks is a direct fallout of Proposition 6.[3]

Corollary 1. *Let Assumption 1 hold and suppose that the matrix of synergies G is symmetric. Then, the optimal profits are given by*

$$\pi_d = \frac{1}{4}\mathbf{1}^T[\beta\mathbf{I} - \gamma G]^{-1}\mathbf{1}$$

which in terms of Katz-Bonacich centralities is

$$\pi_d = \frac{1}{4\beta}\mathbf{1}^T\mathcal{K}\Big(G, \frac{\gamma}{\beta}\Big)$$

Next, we examine what are the differences in profits when the network goes from asymmetric to symmetric. Following Candogan et al. (2012), we consider all profits for the class of adjacency matrixes parametrizes by $\alpha \in [0,1]$, $G^\alpha = \alpha G + (1-\alpha)G^T$. We find that there are no changes in profits as the level of asymmetry of the network fluctuates. Furthermore, these networks give the same optimal contract. This property is highlighted in the next corollary.

Corollary 2. *The family of matrixes $G^\alpha = \alpha G + (1-\alpha)G^T$, $\alpha \in [0,1]$ give the same set of optimal contracts, and thus the same profits.*

Strictly speaking, these matrixes are in the same equivalence class concerning the contracting game proposed. Namely, the matrixes in G^α yield the same optimal contract. Since we know how to find a symmetric network that yields the same solution, this corollary allows us to generalize results that we obtain for the far more tractable symmetric matrixes. Other implications of this corollary and their intuition will be discussed in the following sections.

We conclude this section with a characterization of the upper and lower bound of the profits of the firm using the spectral properties of the network of synergies. The following Proposition provides theses bounds when G is symmetric.

[3]We call symmetric network to an undirected graph, viz. the adjacency matrix is symmetric.

Proposition 7. *Let Assumption 1 hold. Consider a symmetric network structure G, and let λ_{min} and λ_{max} be the smallest and largest eigenvalues of the matrix G. Then, the optimal profits satisfy*

$$\frac{n}{4}\frac{1}{(\beta-\gamma\lambda_{min})} \leq \pi_d \leq \frac{n}{4}\frac{1}{(\beta-\gamma\lambda_{max})}$$

The above result is significant because it showcases that global network characteristics are sufficient to bound the profit. If the range among eigenvalue is small, then the bounds of the profits are closer. These results also allow us to connect our results with a body of literature on aggregate characteristics of networks. The largest eigenvalue relates to averages and maximum degrees[4]. As a consequence, when the average degree falls, the upper bound goes down as well; the same is true for the maximum degree. Following Bramoullé et al. (2014), the smallest eigenvalue is relevant because it captures how much the network magnifies agents' actions. According to Lovász (2007), the smallest eigenvalue measures how much the network is close to being bipartite[5] This last result suggests than when the network is close to being bipartite, the lower bound sinks.

3.2 Efficient Solution under Moral Hazard

In this section, following Hölmstrom (1982), we demonstrate that the efficient solution is implementable under nonobservable effort. The timing of the two-stage game with nonobservable effort is as follows. First, the owner of the firm offers a compensation scheme $\{w_i(y)\}_{i \in N}$. Then, each worker determines his optimal effort simultaneously, maximizing his payoff. Finally, the payoffs of all workers and the owner are realized.

The owner of the firm needs to solve the following program:

[4]The average degree is the number of links per node in the network. The maximum degree is the higher number of connections of any node in the graph.

[5]Bipartite networks are those whose nodes are divided into two sets, and only links between nodes in different sets are allowed.

$$\max_{w_i(y(a))} \; y_g(a) - \sum_i w_i(y_g(a))$$

subject to

$$w_i(y_g(a_i, a_{-i})) - \beta a_i^2 \geq 0, \qquad i \in N \tag{IR}$$

$$a_i^* \in \underset{a_i}{\operatorname{argmax}} \; w_i(y_g(a_i, a_{-i})) - \beta a_i^2, \qquad i \in N \tag{IC}$$

$$w_i(y(a)) \geq 0, \qquad i \in N \tag{LL}$$

We pose the contract scheme in general terms, for both the single wage and the perfect wage discrimination case.

Proposition 8. *In the noncoperative game with nonobservable efforts , there exists a wage scheme that satisfies (IR) such that a^* is an efficient Nash Equilibrium.*

The idea of the proof follows Hölmstrom (1982). A group of penalities that punish all the workers when the realized output is lower than $y * (a^*)$ so that a^* is a Nash equilibrium. The penalties proposed may seem extreme, but for the sake of clarity, are chosen. However, as Hölmstrom remarks, the idea can be implemented with a fixed wage, and a group of bonuses to be rewarded is the target output is achieved.

Therefore, we conclude that under moral hazard, the optimal solution we posed in previous sections can be accomplished. We point out that the workers' strategy of exerting no effort is also a Nash Equilibrium. Therefore, the contract structure presented in the proof is not ideal when coordination is not possible between the workers and the owner of the firm.

Chapter 4

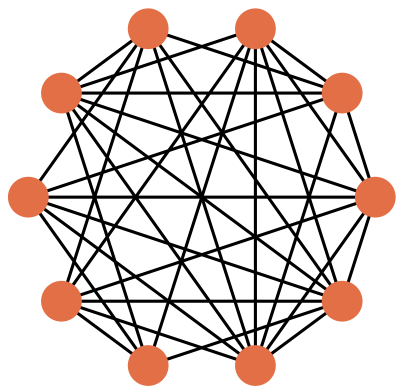
A Numerical Exercise: Incentives and Network Structure Revisited

In this chapter, we intend to understand the differences among the contracting regimes and to comprehend the impact of accounting for network effects. With this in mind, we provide a set of examples and simulations with different network structures. The code is available at *http://github.com/gisslab/teams-network-technologies* (Labrador-Badia, 2020).

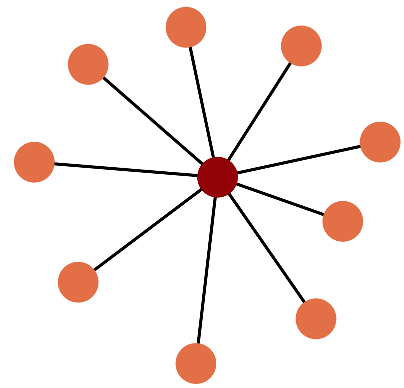
4.1 Comparison Between Single Wage and Perfect Discrimination

In this section, we compare the optimal contract in the case of a single wage with the case of perfect wage discrimination. To better understand the differences between these two settings, we first propose the analysis of two examples. The following figure depicts the network structure employ in these examples. For ease of exposition, we do not show the direction and weight of links.

First, we randomly generate an upper triangular matrix complying with Assumption 1, when



(a) Uniformly random network

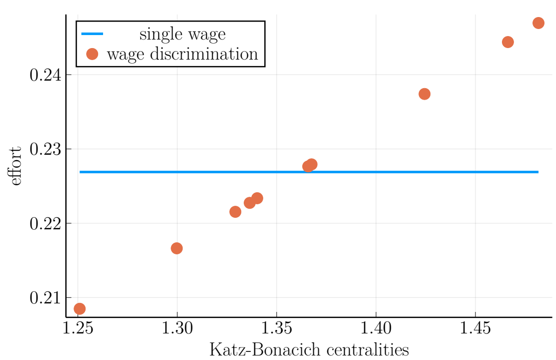


(b) Uniformly random star

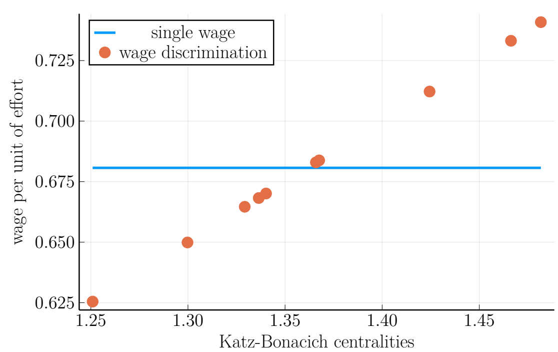
Figure 4.1: Networks with links uniformly generated between 0 and 2.

Source: own elaboration.

$\beta = 3$, $\gamma = 0.25$ and $n = 10$ (Figure 4.1a)[1]. The solution to this example follows from Proposition 1 and Proposition 4. Figure 4.2 depicts the efforts and wages for both the case of single wage and price discrimination considering the random network. In this setup, if the owner of the firm can offer a single salary (blue line), the optimal effort demanded will be $\bar{a} = 0.42$ and the optimal wage will be $\bar{w} = 0.53$.



(a) Optimal effort by Bonacich Centrality



(b) Optimal wage by Bonacich Centrality

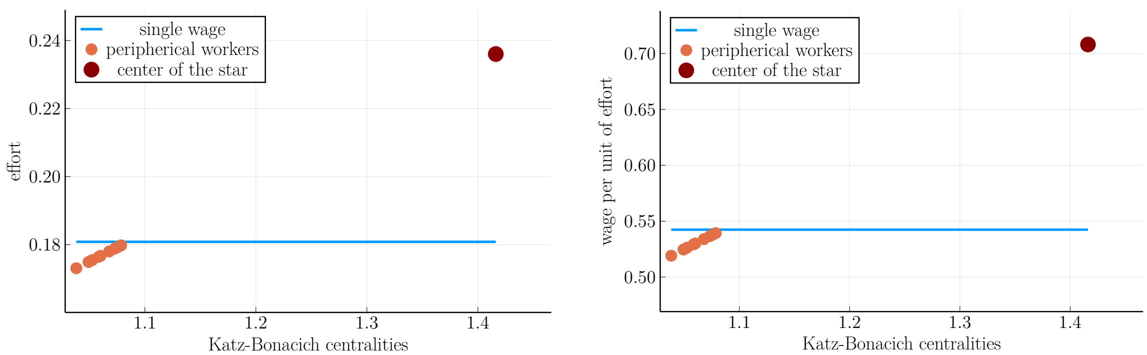
Figure 4.2: Optimal solution for a randomly generated graph: $\beta = 3$, $\gamma = 0.25$ and $n = 10$.

Source: own elaboration.

As predicted by our results, the workers that have greater Katz-Bonacich centrality are the

[1]We generate the weights of the adjacency matrix with uniformly random numbers between 0 and $\frac{\beta}{\gamma * n}$

ones who are offered higher wages. Because these are the most influential agents to the outcome of production, they receive the most favorable wages. The wages and efforts of the case with discrimination fluctuate over the single wage line. Depending on how central a worker is, he is offered a wage lower or higher than the single wage. The profits in the unique wage setup are $\pi_s = 2.10$, and for the case of discrimination of wages are $\pi_d = 2.18$. The firm benefits from being able to internalize the structure of the network. For the no discrimination case, only one metric, the overall strength of synergies, is taken into account in the optimization; this leads to a potential loss in profits.



(a) Optimal effort by Bonacich Centrality

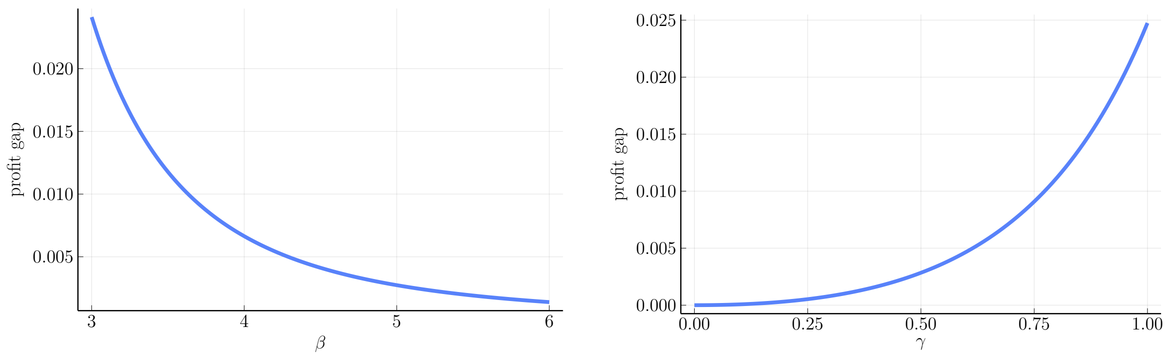
(b) Optimal wage by Bonacich Centrality

Figure 4.3: Optimal solution for a star graph: $\beta = 3$, $\gamma = 0.25$ and $n = 10$.

Source: own elaboration.

The second example is a star graph that represents a team of ten employees (Figure 4.1b). We can see in Figure 4.3 the effort and wage as a function of the centralities. The outlier in the superior right corner is the worker located in the center of the star. This individual has by far the highest centrality, and he gains the most substantial wage. The other nine workers have a closer centrality, and their efforts and compensations are below the single wage line. In contrast, the example of a uniformly random graph the efforts and salaries are evenly split above and below the single wage line. The distinction between these two examples reminds us that even though we have general results, the specific structure of the network is critical for the particular solution.

We now delve into the difference in profits under a single wage regime and a perfect discrimination scheme when other variables of interest change. To accomplish this task, we will focus on the profit gap between the two contract structures, i.e., $\pi_d - \pi_s$. We conduct a set of simulations across various network structures and parameters. In sub-figure 4.4a, we show the gap as a function of the cost of effort obtained through a simulation of 100 different uniformly random matrices. In the Appendixes, we discuss other network structures while β and other exogenous variables change.



(a) Cost of effort

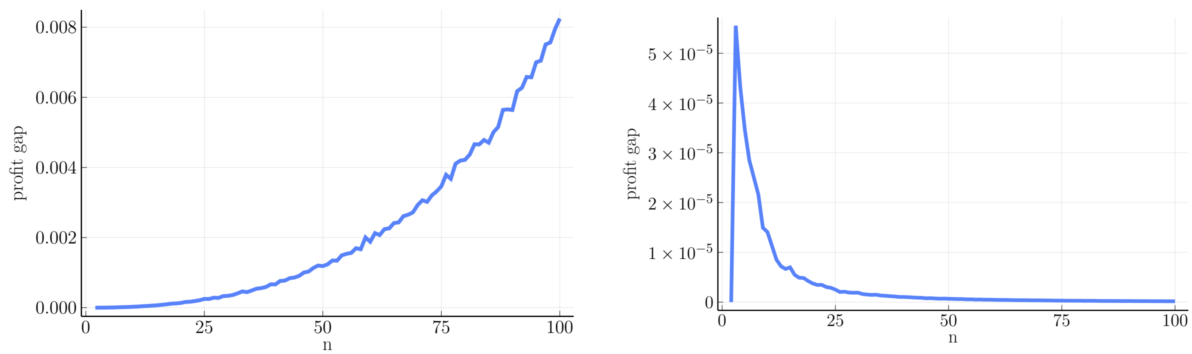
(b) Strength of network effects

Figure 4.4: Profit gap of firms with 50 workers and uniformly random networks.

Source: own elaboration.

Some observations are immediately evident. First, the profits of the discrimination case are higher than the profits of the single wage. It is no surprise: the endogenous space of solution of the single wage contract is a subset of the perfect discrimination space. Intuitively, as we discuss in the first example of the section, the owner is unable to incorporate network features other than the aggregate strength of interactions in the less sophisticated setup, and benefits from assimilating the individual's centrality in the more advanced scheme. Second, the profit gap decreases as the disutility of effort grows. As a robustness test, we check that the profit ratio ($\frac{\pi_s}{\pi_d}$) increases, which points to the same conclusion. Namely, as the disutility of effort raises, the owner of the firm has fewer incentives to discriminate.

Following an analogous analysis, the opposite conclusion is valid for the strength of interaction (Figure 4.4b). The incentives of the firm to discriminate become larger as the network characteristic γ strengthens. Under a positive lens, suppose we observe discrimination in a firm with heterogeneous degrees of interaction among their workers, it might be in part due to a large strength of interaction that impacts production. This line of analysis may also have repercussions in social dimensions. For example, we can make a normative argument for fairness in this context: if the intensity of externalities is small, then the gains from discrimination do not justify the unequal payments.



(a) Number of workers with $\mathbb{E}_{i,j \in N}(g_{ij})$ fixed

(b) Number of workers with $\bar{g}$ fixed

Figure 4.5: Profit gap of uniformly random networks as a function of the number of workers: $\beta = 3$ and $\gamma = 0.25$.

Source: own elaboration.

In figure 4.5, we compile two different approaches in reference to the study of the effect of the number of workers on the gap. The figure on the left shows the difference in profits, when the n increases, ceteris paribus the expected value of weighted links. In other words, as n grows, the strength of interactions between workers stays the same. The relationship between the number of workers and the gap is monotonically increasing under these conditions. In other words, not because the number of workers is proportionally growing and directly bringing extra profits, it is less imperative to make use of the individuals' centrality. The figure on the left depicts the same function, but as $\bar{g}$ stays constant. This comparative static was analyzed in section 3.1.1 and the insights are very similar. The network aggregate metric $\bar{g}$, represents the total sum of weighted links and is a measure of the overall strength of interactions. For a very small number of workers, the relationship is positive between n and the profit gap. However, after a threshold n, the difference between the two contract schemes grows smaller. Assuming $\bar{g}$ constant, the bilateral interactions between workers become weaker, and there are fewer incentives to discriminate according to the position of the network, which is what distinguishes workers in this setting. In both cases, the profits grow, while the gap goes to zero. If hiring more employees damages the peer to peer complementarities, then we should expect that workers become homogeneous and the gains from discrimination dwindle.

4.2 When Does the Network Matter?

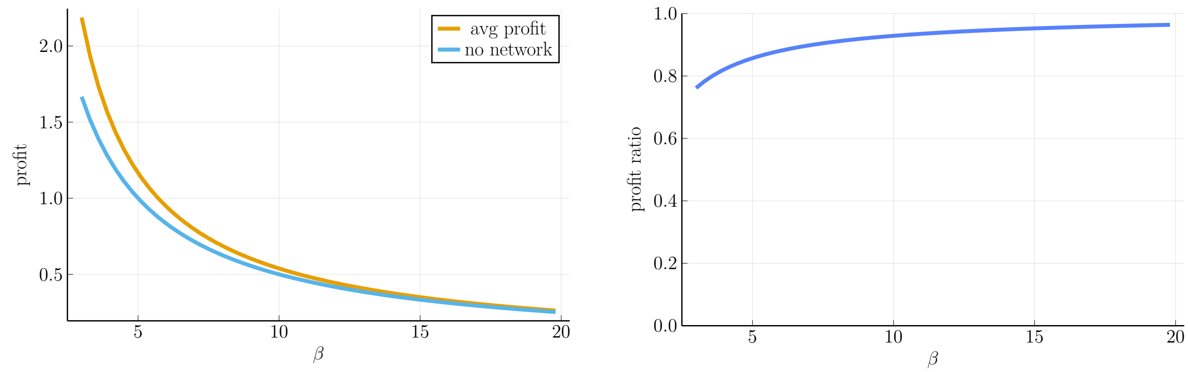
Up to this point, we have supposed that the owner of the firm has perfect knowledge of the network. This information could be difficult or costly for the firm to obtain. Thus, it is highly helpful for the owner of the firm to know in which situations the information about the network structure is most relevant. Answering the previous question is the goal of this section. With this purpose, we contrast the following two scenarios for the case of perfect discrimination of workers:

(i) The owner of the firm only worries for maximizing the sum of aggregate efforts and assumes no network effects are present. In this setting, the workers take into account the externalities of the work of their co-workers.

(ii) The owner of the firm has perfect knowledge of how the synergies affect the outcome of production.

Let π_0 denote the profits in the first case and π_g the profits in the second case. The closed-form expression for the profits in the first setup is given by $\pi_0 = \frac{n}{4\beta}$ and Proposition 6 characterize the profits for the second scenario. The expression of π_0 follows from the objective function of the owner of the firm: $\sum_i a_i - \sum_i w_i(y)$. Because she is only rationalizing the sum of efforts, the threshold output for rewarding workers will be y_0^* without including the complementarities of production. The workers, aware of their work interactions and their impact in production, will adjust their effort to accrue the output y_0^*.

Intending to illustrate the impact of network effects on profits, we provide a series of simulations. First, we analyze the impact of network effects when the cost of exerting effort changes.



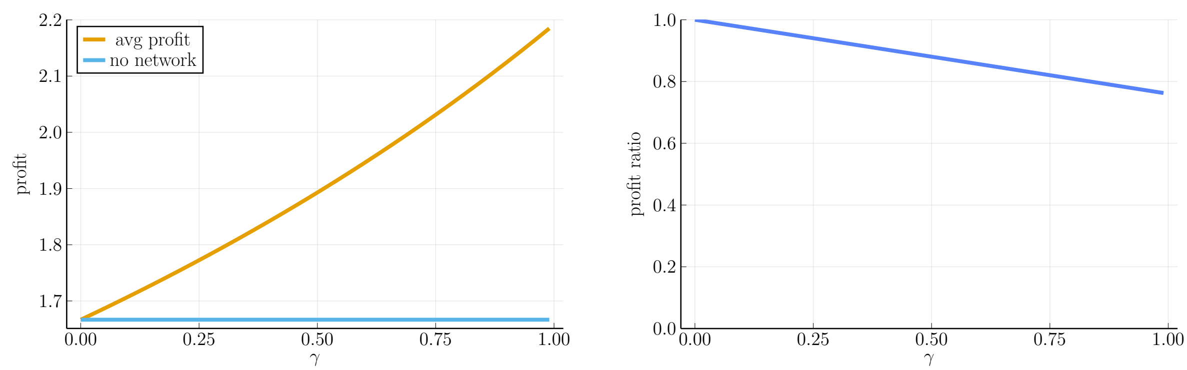
(a) Profits

(b) Profit ratio

Figure 4.6: Profits a function of effort disutility: $\gamma = 0.25$ and $n = 20$.

Source: own elaboration.

In Figure 4.6, we depict both profits as a function of the disutility of effort. Consistent with the closed-form of profits, we observe that in both cases, profits are decreasing in β. Notably, the effect of β through the Katz-Bonacich centralities it is more marked as β grows smaller. For β big enough, the effect is close to zero. This simulation showcase that accounting for network externalities can bring considerable benefits for the firm when the cost of effort is relatively low. Moreover, as we present in Figure 4.6b, the profit ratio confirms that :



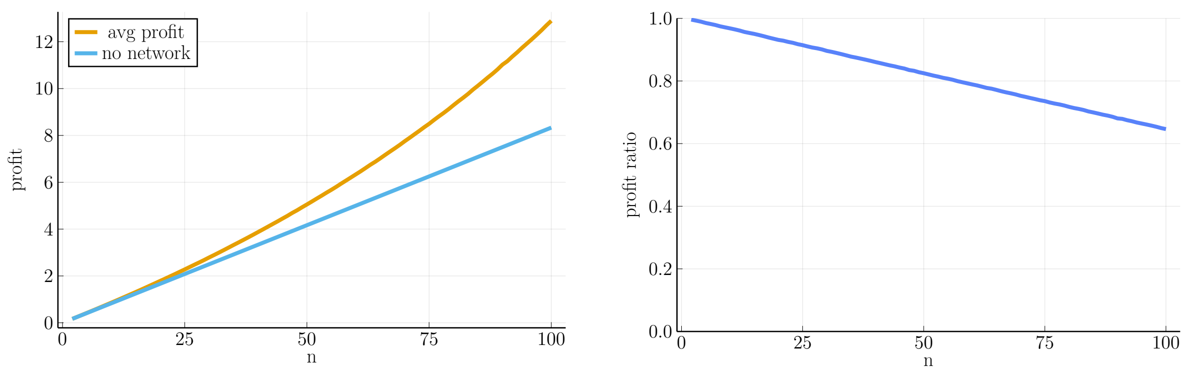
(a) Profits

(b) Profit ratio

Figure 4.7: Profits as a function of intensity of interactions: $\beta = 3$ and $n = 20$.

Source: own elaboration.

In order to understand how the intensity of externalities influence profits, we turn our attention to Figure 4.7. The breakdown of this variable is simple; it is trivial that for π_0, there are no variations given that γ is not present in the owner of the firms reasoning.This steams from the owner of the firm's only goal: maximizing the sum of efforts. Even if the workers account for the synergies, they will internalize the planning of the owner of the firm, and they will aim to the sum of efforts. The effect of γ through the Katz-Bonacich centralities has not been examined until now. We observe that the function of profits increases and does not seem to follow a linear pattern. The conclusion is fairly natural: as the intensity of network effects rises, the benefits of considering the synergies increases. The decreasing profit ratio supports this intuition. This finding confirms that taking network effects into consideration can produce significant gains.



(a) Profits

(b) Profit ratio

Figure 4.8: Profits as a function of number of workers: $\beta = 3$ and $\gamma = 0.25$.

Source: own elaboration.

Now we study how much accounting for network effects matters when the number of workers changes. In both cases, the profits increase when only the number of workers grows. In Figure 4.8, we find the plots for the two cases. The advantages of considering the network structure grow when the number of workers increases. The effect of externalities in production for randomly generated graphs is enhanced when n raises.

Finally, we discuss the implications of Corollary 2 in the context of the importance of network effects. In our simulations, we found that considering the network structure always gave a higher profit, which is intuitive and can be demonstrated from the closed-form expressions of profits. In addition, all networks in the class $G_\alpha = \alpha G + (1-\alpha)G^T$ yield the same profit; this is consistent with the analytic result of the above-mentioned corollary. This outcome differs from some of the previous literature in the following fashion: when the network of bilateral relations is symmetric, there are not benefits of considering the network. In the present set up, because influencing the effort of other workers as being influence by others are equally crucial for production, the effects add up. In contrast, in the works of Candogan et al. (2012) and Jadbabaie and Kakhbod (2019), the trade-off between compensating an agent with many out-links, and the gains in utility from an equal number of in-links, cancels out.

This comparison has similar patters to the discrimination discussion of the previous section. However, the differences between the contracting regimes are minuscule when contrasted with the difference of not accounting for the network. We observe in our simulations that the profit loss is usually more than ten times bigger in the current analysis. Of course, this should not be taken literally, given that our case of no network is an artifice to understand better the consequences of not considering the structure of synergies. Besides, the careful way in which we have chosen our parameters to comply with our assumptions further biased the results. Nonetheless, our results indicate that the incentives of accounting for network effects are undeniable larger than the incentives stem from the changes in contract schemes. Furthermore, the static analysis in these two sections coincides for the variables of interest. Especially, it seems unquestionable that when the network effects are greater, it is more urgent to take greater advantage of the information at hand.

To conclude, we emphasized the importance of discerning the cases in which it is substantial to account for the network. If monitoring the synergies and finding out the network structure implies a high cost, this discussion suggests that firms are more likely to make use of network information when the differences in profits are greater. On the other hand, when the owner of the firm knows the network structure and incorporates this information while contracting, this translates into gains, though sometimes small. Hence, we can predict that organizations that use the information of complementarities in production enjoy higher profits than their pairs.

Chapter 5

Conclusions

This thesis analyzes a multi-agent contracting game with complementarities in the production function. Our main contribution is the inclusion of a sophisticated social structure in the production process that allows us to encompass circumstances where the workplace's synergies stem from productive activity. In contrast with complementarities resulting from the social dynamics that affect workers directly, and production indirectly, externalities in our setting can only indirectly influence workers. The study of this framework provides several intuitive insights into how firms incorporate the intricacies of production technologies.

We focus on the differences between two contracting restrictions: single wage and perfect wage discrimination. We show that in the case of no discrimination, the owner of the firm only uses information about an aggregate measure of the network. When discrimination is allowed, wages and efforts are directly proportional to one particular characteristic of individuals: their Katz-Bonacich centrality. Thus, our results show that this centrality measure arises in contracting frameworks, not only when there are direct externalities in the payoffs of agents (Jadbabaie and Kakhbod (2019); Belhaj and Deroïan (2019)).

We also supply bounds of the profits that depend on aggregate measures of the network (the smallest and largest eigenvalue) for perfect wage discrimination. These bounds can provide directions for experimental and empirical work, especially when the whole network structure is unknown. These results demonstrate that the network effects shape not only individual outcomes but aggregate outcomes as well. We discover that our results are invariable to the level of asymmetry, that is, symmetric networks give the same qualitative results as asymmetric networks. In the framework of a monopolist that sells a network good, Candogan et al. (2012), Bloch and Quérou (2013), and Jadbabaie and Kakhbod (2019) find network irrelevant results for symmetric networks; in contrast, in our setting, the network effects when present are always substantive. Furthermore, we identify the parameters for which the gap between discrimination of wages yields a higher profit for the firm.

We distinguish the profit gap between taking the network structure into account and ignoring network effects. Although wage discrimination is desirable, for the space of parameters that we proposed, the differences between discrimination and no discrimination are rather small; especially, in comparison with the profit gap of not accounting for the network. This point is a compelling argument in favor of no discrimination, given that the incentives to discriminate are small, and the cost of acquiring information about the network of synergies could be considerable.

Finally, we would like to propose some paths for future work. Our results are ideal when coordinating workers is feasible; consequently, we propose to explore solutions that allow coordination of agents, for example, rank-order tournaments. Extending our analysis to nonlinear production technologies is another avenue for future research. We pose synergies in a simple mathematical form to get tractable linear best replies; however, pondering more complex network technologies could yield different results. Introducing uncertainty and interlink individual outputs in the production function could be another exciting direction going forward. One way of undertaking this goal is with a sort of relative performance evaluation that includes noisy signals and local network effects. Concretely, the output of any given worker depends on the output of other workers via the network of synergies. Also, there are other aspects of research on social networks like clustering that would be appealing to draw into this model.

Appendices

Appendix A

Proofs

This appendix assembles all the proofs.

Proof of Proposition 1.

Proof. The optimal efforts and wages follow from clearing equation 3.6.

By summing for all j, the expression from Assumption 1 we get

$$\gamma \sum_j \sum_i g_{ij} < n\beta$$

This is equivalent to

$$n\beta - \gamma\bar{g} > 0$$

This inequality implies two things: 1) the optimal efforts are always positive, and 2) the objective function of equation 3.5 is strictly concave in a.

Therefore, we can guarantee the interior solution is a global maximum of the space where $w > 0$.

Since we are forcing a single wage, we do not contemplate the scenario in which the owner offers some workers wage equal zero. The corner solution, $w = 0$, yields zero profits, and it is smaller than the profit of the interior solution

$$\pi = \frac{n^2}{4(n\beta - \gamma\bar{g})},$$

which is strictly positive. Thus, the optimal solution is the interior solution a_s^* and w_s^*. □

Proof of Lemma 1.

Proof. By Assumption 1, it follows that for all i, $\sum_j (g_{ij} + g_{ji}) < (2\beta)/\gamma$.

One of the implications of the Perron-Frobenius theorem is that the largest eigenvalue (Perron–Frobenius eigenvalue or spectral radius) of a real square matrix A with non-negative entries is bounded above by $\max_i \sum_j a_{ij}$. Then, applying this result to matrix $G + G^T$, we get

$$\lambda_{max} \leq \sum_j (g_{kj} + g_{jk}),$$

where k is the row with the largest sum of their entries.

Hence, applying the first remark to row k, we obtain the inequality $\lambda_{max} < (2\beta)/\gamma$. □

Proof of Proposition 4.

Proof. The optimal vector of efforts is derived by clearing $\mathbf{a}^*$ in equation 3.10:

$$\mathbf{a} = [2\beta\mathbf{I} - \gamma(G + G^T)]^{-1}\mathbf{1}$$

By substituting the optimal effort vector in the participation constraint, we obtain the optimal vector of wages,

$$\mathbf{w} = \beta[2\beta\mathbf{I} - \gamma(G + G^T)]^{-1}\mathbf{1} \circ [2\beta\mathbf{I} - \gamma(G + G^T)]^{-1}\mathbf{1}$$

By Lemma 1, we can guarantee that the matrix is invertible, and the efforts are non-negative numbers (Ballester et al. (2006)). Thus, the limited liability condition is satisfied, i.e., the wages are non-negative. Furthermore, since the Katz-Bonacich centralities are by definition greater or equal than one, the solution $\mathbf{a}^*$, $\mathbf{w}^*$ is the unique interior solution. Hence, if the owner of the firm decides to offer positives salaries to all his workers and thus demand a positive effort, the previous expressions constitute the optimal contract he offers.

Next, we handle corner solutions. Let us assume that the owner of the firm offers positive wages to a subset $B \subset N$. That is, there is at least one worker j with $w_j = 0$. We denote G_B the matrix that only contains the rows and columns associated with individuals in B; likewise, we denote $\mathbf{a}_B$ to the vector of entries of agents in B. Then, the modified optimal condition for the workers in B is

$$\mathbf{1} + \gamma(G_B + G_B^T)\mathbf{a_B} - 2\beta\mathbf{a_B} = 0 \tag{A.1}$$

which implies that the efforts and wages have the same form as the interior solutions.

Now, we check that the matrix $2\beta\mathbf{I_B} - \gamma(G_B + G_B^T)$ is non-singular, and the elements of the vector are non-negative. We denote $\lambda_{max}(A)$ the maximum eigenvalue of matrix A. By Debreu and Herstein (1953)'s Theorem 1, the largest eigenvalue of $(G_B + G_B^T)$ complies with

$$\frac{2\beta}{\gamma} > \lambda_{max}(G + G^T) > \lambda_{max}(G_B + G_B^T),$$

which means by the arguments exposed before that $2\beta\mathbf{I_B} - \gamma(G_B + G_B^T)$ is invertible and non-negative.

Finally, we verify that the expressions for a_i and w_i constitute a maximum of the objective function for the interior solution. The second-order condition establishes that the following matrix must be definitive negative to guarantee that a^* is a maximum:

$$H = \gamma(G + G^T) - 2\beta\mathbf{I}$$

First, notice that $\gamma(G + G^T)$ is a symmetric matrix and because $2\beta\mathbf{I}$ is the diagonal matrix

$D(2\beta\mathbf{1})$ is also symmetric. Hence, the matrix H is symmetric as well. A sufficient condition for H to be negative definite is that for all $i \in N$, $2\beta > \gamma \sum_i (g_{ij} + g_{ji})$[1] This condition is met by Assumption 1. The later inequality proves that $-H$ is definite positive; therefore, the program is concave. □

Proof of Corollary 1.

Proof. We first note that the profits as a function of effort in matrix form can be written as

$$\pi_d = \mathbf{a}^T[\mathbf{1} + (\gamma G - \beta \mathbf{I})\mathbf{a}]$$

When G is a symmetric matrix, $G + G^T = 2G$. It follows that $a^* = \frac{1}{2}[\beta\mathbf{I} - \gamma G]^{-1}\mathbf{1}$. Then:

$$\begin{aligned}
\pi_d &= \frac{1}{2}[[\beta\mathbf{I} - \gamma G]^{-1}\mathbf{1}]^T[\mathbf{1} - \frac{1}{2}(\gamma G - \beta\mathbf{I})[\gamma G - \beta\mathbf{I}]^{-1}\mathbf{1} \\
&= \frac{1}{2}[\beta\mathbf{I} - \gamma G]^{-1}\mathbf{1}]^T[\mathbf{1} - \frac{1}{2}\mathbf{1}] \\
&= \frac{1}{4}\mathbf{1}^T[[\beta\mathbf{I} - \gamma G]^{-1}]\mathbf{1}
\end{aligned}$$

and factorizing β out of the inverse matrix gives $\pi_d = \frac{1}{4\beta}\mathbf{1}^T\mathcal{K}(G, \frac{\gamma}{\beta})$ □

Proof of Corollary 2

Proof. From equation 3.8 the program is

$$V(\mathbf{a}) = \mathbf{1}^T\mathbf{a} + \mathbf{a}^T G\mathbf{a} + \beta\mathbf{a} \circ \mathbf{a}\mathbf{1}$$

We need to demonstrate that for all matrixes in G^α, $\alpha \in [0, 1]$ the program is the same, and

[1] If a square matrix is (strictly) diagonally dominant with real nonnegative elements in the diagonal it is positive (definite) semidefinite.

the assumptions hold.

For all $\alpha \in [0, 1]$ is true that

$$\begin{aligned}\mathbf{a}^t G^\alpha \mathbf{a} &= \mathbf{a}^t[\alpha G + (1-\alpha)G^T]\mathbf{a} \\ &= \alpha(\mathbf{a}^T G\mathbf{a}) + (1-\alpha)(\mathbf{a}^T G^T \mathbf{a})\end{aligned}$$

Note that because $(\mathbf{a}^T G\mathbf{a}) = (\mathbf{a}^T G^T \mathbf{a})$, for all $\alpha \in [0, 1]$:

$$(\mathbf{a}^T G^\alpha \mathbf{a}) = (\mathbf{a}^T G\mathbf{a})$$

The eigenvalues of the symmetrized network are the same for all $\alpha \in [0, 1]$, then the proof in Proposition 4 holds, and the problems are equivalent. □

Proof of Proposition 7

Proof. Let us recall the expression for the profit as a function of effort in matrix form:

$$\pi_d = \mathbf{1}^T\mathbf{a} + \gamma\mathbf{a}^T G\mathbf{a} - \beta\mathbf{a}^T\mathbf{a}$$

For this proof we will rewrite the Bonacich centrality vector of a matrix A in terms of the centrality matrix, that is, $\mathcal{K}(A, \delta) = K(A, \delta)\mathbf{1}$ where $K(A, \delta) = [I - \delta A]^{-1}$. In those terms the optimal effort of Proposition 4 becomes

$$\mathbf{a} = \frac{1}{2\beta}K(G + G^T, \delta)\mathbf{1}$$

where $\delta \equiv \frac{\gamma}{2\beta}$ is the discount factor. For clarity from now on we denote $K \equiv K(G + G^T, \delta)$. Then, the optimal profits turn into

$$
\begin{aligned}
\pi_d &= \frac{1}{2\beta}\mathbf{1}^T K\mathbf{1} + \frac{\gamma}{4\beta^2}(K\mathbf{1})^T GK\mathbf{1} - \frac{1}{4\beta}(K\mathbf{1})^T K\mathbf{1} \\
&= \frac{1}{2\beta}\mathbf{1}^T \Big[K\mathbf{1} + \frac{\gamma}{2\beta}K^T GK\mathbf{1} - \frac{1}{2}K^T K\mathbf{1}\Big] \\
&= \frac{1}{2\beta}\mathbf{1}^T \Big[K + \delta K^T GK - \frac{1}{2}K^T K\Big]\mathbf{1} \qquad \text{(A.2)}
\end{aligned}
$$

Since G is symmetric, it is a diagonalizable matrix; i.e., there exists an invertible matrix P composed with eigenvectors of G, and a matrix Λ such that it contains the eigenvalues of G in the diagonal($\Lambda = D(eigen(G))$) so that $G = P\Lambda P^{-1}$. We chose P such that their eigenvector is an orthonormal basis, i.e., $P^{-1} = P^T$. Furthermore, because G is symmetric, all the eigenvalues are distinct and real numbers, and $G = G + G^T = 2G$.

We now use the Taylor power series and the previous remarks to rewrite K:

$$K = \sum_{k=0}^{\infty}[\delta(G + G^T)]^k = \sum_{k=0}^{\infty}(\delta 2G)^k$$

By substituting the diagonalization of G, we obtain

$$K = \sum_{k=0}^{\infty}(2\delta P\Lambda P^{-1})^k = \sum_{k=0}^{\infty}(2\delta)^k(P\Lambda P^{-1})^k$$

It is easy to check that $(P\Lambda P^{-1})^k = P\Lambda^k P^{-1}$, and then:

$$K = \sum_{k=0}^{\infty}(2\delta)^k P\Lambda^k P^{-1} = P\Big[\sum_{k=0}^{\infty}(2\delta\Lambda)^k\Big]P^{-1}$$

If we pay attention to the elements in the diagonal matrix $2\delta\lambda$, we see that in the ith position it has $2\delta\lambda_i$, where λ_i is an eigenvalue of G. We can prove that $2\delta\lambda_i < 1$. By Lemma 1 and since the eigenvalues of $G + G^T$ double the eigenvalues of G because G is symmetric, it follows that $2\lambda_i < \frac{2\beta}{\gamma}$ for all i. Thus,

$$1 > 2\frac{\gamma\lambda_i}{2\beta} = 2\delta\lambda_i$$

We can employ the geometric series convergence result in the elements of the diagonal matrix $D \equiv \sum_{k=0}^{\infty}(2\delta\Lambda)^k$. Therefore, the ith element of the diagonal of D is

$$\sum_{k=0}^{\infty}(2\delta\lambda_i)^k = \frac{1}{1-2\delta\lambda_i}$$

The evolution of K summarized in $K = PDP^{-1}$

Using $K = PDP^{-1}$, $G = P\Lambda P^{-1}$ and the orthonormality of P in equation (A.2) we obtain

$$\begin{aligned}
\pi_d &= \frac{1}{2\beta}\mathbf{1}^T\left[PDP^{-1} + \delta(PDP^{-1})^T(PDP^{-1})(PDP^{-1}) - \frac{1}{2}(PDP^{-1})^T(PDP^{-1})\right]\mathbf{1} \\
&= \frac{1}{2\beta}\mathbf{1}^T\left[PDP^{-1} + \delta PD^2\Lambda P^{-1} - \frac{1}{2}PD^2P^{-1})\right]\mathbf{1} \\
&= \frac{1}{2\beta}\mathbf{1}^T\left[P(D + \delta D^2\Lambda - \frac{1}{2}D^2)P^{-1}\right]\mathbf{1} \\
&= \frac{1}{2\beta}\mathbf{1}^T\left[PLP^{-1}\right]\mathbf{1}
\end{aligned}$$

where $L \equiv D + \delta D^2\Lambda - \frac{1}{2}D^2$.

Recall that the eigenvalues of a square diagonal matrix are the elements on the diagonal. The matrix L is the sum of other diagonal matrixes and thus its a diagonal itself. Hence, the eigenvalues of L are given by the following function of the eigenvalues of G ($\lambda = \{\lambda_1, ..., \lambda_n\}$)

$$\begin{aligned}
\phi(\lambda_i) &= \frac{1}{1-2\beta\lambda_i} + \frac{\delta\lambda_i}{(1-2\beta\lambda_i)^2} - \frac{1}{2(1-2\beta\lambda_i)^2} \\
&= \frac{1-2\beta\lambda_i + \delta\lambda_i - \frac{1}{2}}{(1-2\beta\lambda_i)^2} \\
&= \frac{1}{2}\frac{1}{(1-2\beta\lambda_i)}
\end{aligned}$$

In short, the elements of the diagonal matrix L are given by $\{\phi(\lambda_i)\}_{i\in N}$, where ϕ is non decreasing in λ_i for all i and positive by Lemma 1.

Using the following lemma taken from Jadbabaie and Kakhbod (2019), we obtain the bounds of the profit.

Lemma 2. ***Jadbabaie and Kakhbod (2019) (Lemma 3.)***
Let μ_{min} and μ_{max} be the smallest and the largest eigenvalue of the square matrix M(with n distinct eigenvalues). Then

$$\mu_{min}\mathbf{x}^T\mathbf{x} \leq \mathbf{x}^T M\mathbf{x} \leq \mu_{max}\mathbf{x}^T\mathbf{x}$$

Notice that the smallest and largest eigenvalue of PLP^{-1} are ϕ_{min} and ϕ_{max}. This follows from ϕ being monotonically increasing in λ_i. Employing the lemma previously enunciated on PLP^{-1} in the expression of π_d we have that

$$\frac{1}{2\beta}\phi(\lambda_{min})\mathbf{1}^T\mathbf{1} \leq \frac{1}{2\beta}\mathbf{1}^T\left(PLP^{-1}\right)\mathbf{1} \leq \frac{1}{2\beta}\phi(\lambda_{max})\mathbf{1}^T\mathbf{1}$$

By substituting $\psi(\lambda_i)$ and δ, and after some manipulation we obtain the bounds for π_d

$$\frac{n}{4}\frac{1}{(\beta-\gamma\lambda_{min})} \leq \pi_d \leq \frac{n}{4}\frac{1}{(\beta-\gamma\lambda_{max})}$$

The proof is now complete. □

Proof of Proposition 8

Proof. Suppose the synergies are accounted for in the production function. The proof is very similar to the one in Hölmstrom's paper. There is a group of penalties that are enough to police agent's effort[2],

$$w_i(y) = \begin{cases} w_i^* & y \geq y(a^*) \\ 0 & y < y(a^*) \end{cases}$$

where w_i^* is the efficient solution of previous sections.

[2]First noted by Mirrlees (1974) for a single agent.

Recall that $w^* \geq \psi_i(a^*) > 0$, so the scheme proposed by construction complies with limited liability. The Budget Condition holds, which is always true when there are synergies on the production function and Assumption 1 holds.

Assume all the workers play the efficient effort a^*. We need to verify that the actions a^* are incentive compatible. For all i, the best response to a^*_{-i} and $w_i(y)$ is to exert a^*_i. The worker has no incentive to deviate: if he lowers his effort, the threshold production is not attained, and if he increases his effort, he will gain the same wage and will have a higher disutility from the effort. Notice that he is indifferent between exerting 0 and exerting a^*; we assume that in these circumstances, he chooses a^*. The wage scheme in a^* satisfies the participation constraint (IR) due to $k_i - \psi_i(a^*) > 0$. □

Appendix B

Network Structures

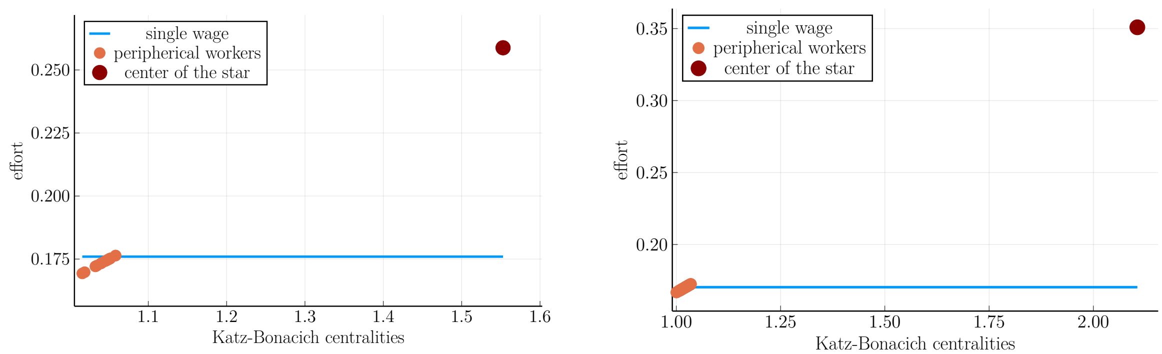
To illustrate the generality of our setting on the underlying network and to comprehend the impact of the different architectures on our results, we analyze some of the most studied network structures.

Star networks

Figure B.1 display two simulations of 20 nodes and 100 nodes, respectively, of a random uniform star. In the star networks, the differences between Katz-Bonacich centralities are small among the peripherical agent. The differences in salaries between the single wage and the central worker grow as the number of workers increases from 20 to 100. That is because the center of the star strengthens his centrality as the number of workers grows; this increment is not reflected in the single wage regime.

Line networks

For a line network, the differences between Katz-Bonacich centralities are small. The two extremes of the line network are the ones depicted in dark red, and the central points are the ones depicted in color green; see Figure B.2. The extremes of the line are way below the single wage line. When the number of workers increases, the single wage rises with respect to the case of 20



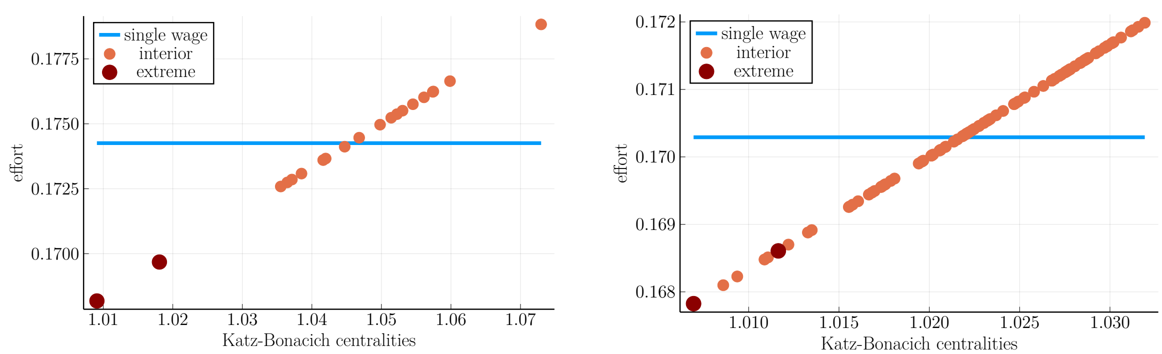
(a) $n = 20$

(b) $n = 100$

Figure B.1: Optimal effort of a star networks: $\beta = 3$ and $\gamma = 0.25$.

Source: own elaboration.

workers. The increase is not as large as in other structures, e. g. star, preferential attachment, and is instead positive. That is because the "special" points have lower centrality in the case of the line, and higher centrality in the case of the star.



(a) $n = 20$

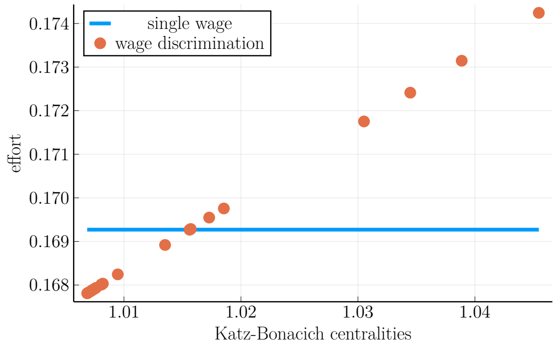
(b) $n = 100$

Figure B.2: Optimal effort for a line network: $\beta = 3$ and $\gamma = 0.25$.

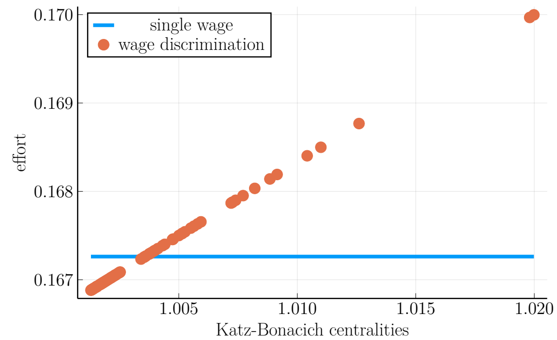
Source: own elaboration.

Preferential attachment

The dynamic model known as preferential attachment was proposed by Barabási and Albert (1999). Figure B.3 depicts simulations of 20 nodes and 100 nodes of preferential attachment networks starting with one node. In these networks, there are a few "hubs" highly connected, which are the ones with the higher centralities and thus wages. When the number of workers rises, the relative difference between the single wage line and the highest-paid worker enlarges.



(a) $n = 20$



(b) $n = 100$

Figure B.3: Optimal efforts for a preferential attachment graph: $\beta = 3$ and $\gamma = 0.25$.

Source: own elaboration.

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