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COALITION FORMATION WITH HETEROGENEOUS AGENTS

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QUE PARA OBTENER EL GRADO DE

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PRESENTA

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A mis padres, Luz y Cuauhtémoc, mis pilares, mi ejemplo y mi meta,
a mi hermana Andrea, mi compañera y mejor amiga,
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Abstract

When heterogeneous individuals form groups in order to carry out productive activities, equal sharing of surplus implies to loss of efficiency due the trade-off between size or homogeneity each coalition faces. Such loss in efficiency can be mitigated by deviating from equal sharing, e.g., the proportional sharing rule. We show that, under proportional sharing rule, there is a unique stable and efficient coalition structure, which is the grand coalition. We then find conditions under which smaller coalitions can form a constrained efficient and stable coalition structure. We show that such exogenous bounds on coalition size can be endogenized by introducing individualized expansion costs. When such costs implies proportional cost sharing, there is an efficient and stable structure. Finally, we consider a convex combination of the equal and proportional sharing rules, under which individual optimum size may vary non-monotonically with respect to ability.

Keywords: Coalition formation, Profit sharing rules.

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Chapter 1

Introduction

In many situations, individuals decide to form groups in order to exploit the benefits from co-operative works. When a group is formed and the group members together produce a 'surplus', they must also decide how to divide the surplus among the participants. The 'surplus sharing rule' can be exogenously given or it may be endogenously determined. When individuals intend to form coalitions, an exogenously given sharing rule crucially influences an individual's decision about which group to belong to. This in turn determines the formation of stable coalitions. In the present work, we focus on fixed surplus sharing rules, and their role in group formation.

The *equal sharing rule* is a popular way to divide surplus among group members even when individuals contribute unequally. Such rules are simple to design and implement. They arguably reduce the organizational costs of designing more complex reward mechanisms and may prevent rent-seeking behavior. Equal sharing rules also obeys the criterion of fairness and respond to social conventions. In many academic disciplines, coauthors equally share credits even if they have asymmetric skill levels. Equal division of earnings often apply to partners of law firms or to a group of fishermen. Farrell & Scotchmer (1988) analyze how equal sharing rules affect stability and efficiency of coalitions when individuals are heterogeneous with respect to their productivities. They find that the game of coalition formation under an equal sharing rule leads to a unique equilibrium [stable] partition of the players into coalitions. A stable partition is defined as the partition structure where no new coalition of any size could form, and make all its members strictly better off.

However, the equilibrium partition under the equal sharing rule is inefficient because it induces individuals to face a trade-off between ability and size. On the one hand, larger groups produce more surplus, on the other, forming groups with less able individuals implies lower surplus. In particular, because the sharing rule force group members to share equally the coalitional gains, having less able partners is equivalent to subsidizing them. Differences in ability of the individuals cause the average ability in a group to decrease with the size of the group. Such loss of efficiency that emerges from the equal sharing of surplus among heterogeneous group members can be mitigated by using a different sharing rule. Thus, the main objective of our work is to analyze surplus sharing rules, which are distinct from the equal sharing rule, in enhancing efficiency of coalition formation

We first consider coalition formation with heterogeneous agents under the proportional sharing rule. This rule ameliorates the loss of efficiency that emerges under the equal sharing rule. However, the unique efficient allocation is the grand coalition. In a bid to achieve a stable coalition structure in which smaller groups may be formed, we artificially restrict the coalitions to a limit size. The stable structure turns out to be constrained efficient. We next introduce costs of expanding the groups, which allows us to achieve bounded coalitions endogenously. Finally, we analyze stable partitions by departing from both the equal and proportional sharing rules in that we consider a convex combination of these two rules. In this case, we find that the optimal choice of group size of the individuals is in general non-monotonic in ability.

1.1 Related literature

Farrell & Scotchmer (1988) analyze the role of equal surplus sharing on the formation of coalitions among heterogeneous agents. They analyze conditions under which a unique stable partition exists. In a stable coalition structure, groups are *consecutive*, i.e., all the members of a group have abilities in a given interval. Moreover, when ability is uniformly distributed, more productive individuals form larger groups. However, as we have explained earlier, equal sharing of surplus among heterogeneous individuals lead to inefficiency. They show that the groups in the core partition tend to be inefficiently small due to the agents' preferences for homogeneity. The present work builds on Farrell & Scotchmer (1988) in what we consider a non-equal sharing rule (in particular, the proportional sharing rule) in a bid to restore efficiency. Under proportional sharing rule, the grand coalition is the unique stable coalition structure. In our framework, each group can be very heterogeneous as no individual faces a trade-off between size and homogeneity.

Pycia (2012) analyzes a more general model of sharing rules and characterizes the rules that induce stable partitions (coalition structures). In particular, he finds that a necessary and sufficient condition for stability is that the rule must induce pairwise aligned preferences over coalitions for each level of output. Pairwise alignment condition means that any two agents rank coalitions that contain both of them in the same way. Both the equal and proportional sharing rules meet this condition. In order to study qualitative properties of stable coalition structures induced by such sharing rules, Pycia (2012) shows that a pairwise-aligned rule can be represented by a profile of agents' bargaining functions. In a stable structure, coalitions are formed by agents with similar bargaining functions. However, this induces a loss of efficiency since some beneficial coalitions may not form. This setting also allows one to compare stable coalition structures that are formed with different stability-inducing sharing rules.

We consider two separate cases—the first where the set of agents is finite, and then, a continuum. The stable or 'core' allocation of a coalitional game with continuum of agents resembles the concept of f-core analyzed by Kaneko & Wooders (1982, 1986). The broad idea behind the concept of finite-core (f-core) is that, in a continuum economy, each individual has measure zero; however, the action of each individual may have non-null impact on the overall game, and thus such individuals can be, loosely speaking, considered to be of positive measure. We later focus on endogenous size restrictions by introducing costs of forming coalitions, which is increasing in the group size.

The present work is also related to the literature on *hedonic games* (e.g. Banerjee, Konishi, & Sönmez, 2001; Bogomolnaia & Jackson, 2002). Hedonic games are the games of coalition formation where the payoff accruing to each member of a coalition is completely determined by the characteristics of the members of the she belongs to. With heterogenous individuals, any fixed sharing rule means that the payoff to any agent depends only on the characteristics of her partners in the group, and hence, gives rise to a hedonic coalitional game. In fact Banerjee, Konishi, & Sönmez (2001) introduce the *top-coalition property*, which is a weakening of the *common ranking property* of Farrell & Scotchmer (1988), which is used to guarantee the existence of a core partition.[1] In our model under the proportional sharing rule, such properties are not required as the grand coalition is the only efficient coalition, which always exists.

1.2 Review of Farrell and Scotchmer (1988)

In this section, we provide a brief review of the most important aspects of the model of Farrell & Scotchmer (1988). Each agent i posses a level of ability or talent x_i, which is an individualized productive asset. The payoff of a coalition or group S is the sum of the abilities of all its members, $\sum_S x_i$ multiplied by a factor $t(|S|)$ that reflects the advantages of scale economies. This factor depends only on the size but not on the composition of the coalition S, and is a strictly increasing function. In other words, productive complementarities are *anonymous*.

Under equal sharing rule, the members of a coalition divide equally the aggregate surplus $t(|S|)\sum_S x_i$ among themselves. Therefore, the individual payoff in any coalition S is simply

$$a(|S|) = \frac{t(|S|)\sum_S x_i}{|S|},$$

[1] The common ranking property requires that there is a linear ordering over all coalitions which coincides with any player's preference ordering over coalitions to which she belongs. On the other hand, the top-coalition property requires that for any non-empty subset S of players, one can find a sub-coalition T of S such that all members of T prefer T to any other coalition that consists of some or all members of S.

which is the average surplus of the group. Notice that, because the individuals differ in ability, their contributions will decrease when ability decreases, this means that enlarging a group will eventually leads to lower marginal contributions, and therefore, to lower individual payoff as gains must be divided equally. Formally, $t(|S|)/|S|$ is strictly decreasing beyond a critical group size.

Farrell & Scotchmer (1988) proves the existence and uniqueness of a stable coalition structure, which relies on the following argument. Assume a function $u(S)$ that is a monotonic transformation of $a(|S|)$. Clearly, $u_i(S) = u(S)$ for all $i \in S$ and any coalition S. Thus, the function $u(S)$ allows each individual to rank coalitions. Then, the formation of the stable (or core) partition goes as follows: first, the best coalition (the one that has a higher evaluation under $u(\cdot)$) forms, then this process repeats for the rest of the agents to form the second best coalition, and so on, until the last coalition forms with the leftovers. Notice that this coalition is stable because there is no other coalition that blocks it (i.e., no new coalition can form and makes all it members better off). No member of the best coalition would want to join any other coalition, neither would accept any other member in his group; this argument repeats with the second best coalition and so on. This construction of the equilibrium allows us to know which partitions will form. Furthermore, this partition is generically unique because $u(S)$ or any monotonic transformation of it preserves the same common ranking property.

Next, the authors consider a continuum version of the model where abilities x are distributed according to the distribution function $F(x)$ on the interval $[0,\ M]$. The payment of each individual in the coalition S is thus given by:

$$a(|S|) = \frac{t(|S|)}{|S|}\int_S xdF(x).$$

One important property of the equilibrium is the consecutiveness of its coalitions, which comes out from the way a stable partition is constructed—the best coalition forms with the most able individuals, down to some cutoff. Then, this process repeats itself which chooses the most able individuals (down to some cutoff) from those who are left to form the second best coalition, and so on. Therefore, each coalition consists of individuals with abilities lying in some interval.

Another characteristic of the equilibrium is that each individual gets at least as much utility as a less able individual: indeed, if it were not the case, the individual could take the place of the less able and improve both her utility and the utility of her new group. However, this would contradict the notion of stability. Finally, when the distribution of abilities is uniform, the size of equilibrium coalitions increases with ability, which means that the coalitions of more able individuals are larger. This result follows from the fact that the difference in ability between the least able member and the mean of abilities becomes proportionally less important as the mean increases. In other words, the contribution of the least able member of a high ability coalition is higher, which encourages this coalition to expand.

Chapter 2

The model

2.1 The proportional sharing rule

Consider a finite set of agents $\mathcal{N} := \{1, \ldots, N\}$. An individual $i \in \mathcal{N}$ has ability x_i. According to the *proportional sharing rule*, the payoff to an agent i is given by:

$$u_i(|S|) = t(|S|)x_i,$$

where $t : \mathcal{N} \longrightarrow \mathbf{R}^+$ is a strictly increasing function, and $|S|$ is the size of the coalition that agent i belongs to. The aggregate surplus of coalition S is equal to $t(|S|) \sum_{i \in S} x_i$. Therefore, the aggregate surplus of a given coalition is divided among its members in a way that each agent receives a pay proportional to her productivity.

A coalition structure is a partition $\mathcal{P}$ of the set of agents $\mathcal{N}$. From now on, we shall say an individual is better off under the partition $\mathcal{P}$ than under the partition $\mathcal{Q}$ if she gets higher pay under the former relative to the latter. Likewise, we shall refer as worse off for the opposite situation.

Definition 2.1. *Given two partitions $\mathcal{P}$ and $\mathcal{Q}$, we define the improvement rate $I_{\mathcal{P},\mathcal{Q}}$ of $\mathcal{P}$ respect to $\mathcal{Q}$ as the number of individuals that are better off under $\mathcal{P}$ minus the number of individuals* *that are worse off.*

Let us now introduce the notion of *blocking* by a single or a group of individuals.

Definition 2.2. *Partition $\mathcal{P}$ weakly blocks partition $\mathcal{Q}$ if and only if improvement rate of $\mathcal{P}$ respect to $\mathcal{Q}$, $I_{\mathcal{P},\mathcal{Q}}$, is non negative. We say $\mathcal{P}$ blocks $\mathcal{Q}$ if and only if $\mathcal{P}$ weakly blocks $\mathcal{Q}$ but $\mathcal{Q}$ does not weakly block $\mathcal{P}$. In other words, if $I_{\mathcal{P},\mathcal{Q}}$ is strictly positive.*

The weak blocking relation is a preorder, i.e., it is reflexive and transitive. The [strong] blocking relation is derived from the weak notion of blocking. Note that the case where there is a coalition that guarantees higher payoffs for some individuals without reducing the pays of any of its member is a particular case of the previous definition.

Definition 2.3. *We say a partition $\mathcal{P}$ is stable if there is no other partition that blocks it. In this sense, a stable partition is a maximal of the set of partitions, induced by the weak blocking relation.*

2.2 Efficiency with finitely many agents

Note that, under both the proportional and equal sharing rules, aggregate payoffs are maximized when the grand coalition (which is the trivial partition) is formed. To see this, consider any partition $\mathcal{P} = \{P_1, \ldots, P_h\}$ of $\mathcal{N}$. Then, the total payoffs under this partition are strictly lower than that under the grand coalition:

$$t(|P_1|)\sum_{P_1} x_i + \ldots + t(|P_h|)\sum_{P_h} x_i < t(N)\sum_{P_1} x_i + \ldots + t(N)\sum_{P_h} x_i = t(N)\sum_{i=1}^{N} x_i.$$

Proposition 2.1. *The grand coalition is the unique stable and efficient partition under the proportional sharing rule.*

Unlike the equal sharing rule, under the proportional rule, the grand coalition maximizes individual payoffs because each individual $i \in \mathcal{N}$ obtains $t(N)x_i$ which is higher than what would be her payoff in other coalition structure, i.e., $t(|P_h|)x_i$ for $P_h \in \mathcal{P}$. Thus, the grand coalition is the unique stable partition because there is no other partition that makes an agent better off. Moreover, the grand coalition is efficient because it maximizes the aggregate payoffs. As we have mentioned earlier, under the equal sharing rule, there exists a generically unique stable partition $\mathcal{S} = \{S_1, \ldots, S_j\}$ which consists of coalitions of inefficient sizes because the aggregate payoffs achieved are lower than that under the grand coalition.

2.3 Stability with bounded coalitions

In this section, we shall analyze situations where there are limits to the formation of the grand coalition. We consider two cases. First, we impose exogenous restrictions on the size of each coalition, and analyze the efficiency properties of a stable coalition structure. Second, we endogenize such size restrictions.

2.3.1 Exogenous bounds on coalition size

Let us consider a coalition formation game in which the size or cardinality of each group is bounded above by a positive real number k. Formally,

Definition 2.4. *We say a partition $\mathcal{P}$ is k-feasible if $|P_h| \leq k$ for all $P_h \in \mathcal{P}$.*

Further, we say that a k-feasible partition $\mathcal{P}$ blocks a k-feasible partition $\mathcal{Q}$ if the set of individuals that are better off in partition $\mathcal{P}$ is larger than the set of the individuals that are worse off (or, equivalently, if the improvement index is strictly positive). Thus, a partition is k-stable if there is no other k-feasible partition that blocks it. The following result shows that under the proportional sharing rule, the agents get to exploit the benefits of scale economies because they do not face the trade-off between the size of their coalition and homogeneity (as it occurs under the equal sharing rule). The k-stable partitions under the proportional sharing rule are constrained efficient in the sense that they allow the individuals to exploit scale economies, given the feasibility condition.

Proposition 2.2. *A partition is k-stable, which we denote by $\mathcal{P}_{k^*}$, if and only if it has at most one group of size strictly less than k.*

Proof. First, note that a $\mathcal{P}_{k^*}$ is stable because no coalition of size k can accept more individuals as its members, and any deviation from a partition of size k to that of size less than k would have, at most, an improvement index zero.

For the other implication, we proceed by contradiction. Suppose $\mathcal{P}$ is a k-stable partition and that it has two coalitions P_h and P_l, such that $|P_h|, |P_l| < k$. We show that the payoffs accruing to the individuals of both these coalitions are higher when they form a new coalition $P_h \cup P_l$. We have the following two possible cases:

Case 1: If $|P_h|+|P_l| \leq k$, each individual i in the coalition P_h obtains the payment $t(|P_h|)x_i$, and each individual j in the coalition P_l receives $t(|P_l|)x_j$. Clearly, both coalitions can be merged into one coalition, $P_h \cup P_l$ without violating the k-feasibility condition. Moreover, each member i of the new coalition, $P_h \cup P_l$ would receive the payment $t(|P_h| + |P_l|)x_i$, strictly higher than what he would have obtained by being in either P_h or P_l. Also, notice that by doing so, no individual will be worse off. This is, the partition $\mathcal{P}'$ obtained from $\mathcal{P}$ by merging the coalitions P_h and P_l together would block $\mathcal{P}$, since it makes k individuals better off without reducing the payments of any other individual. Therefore, $\mathcal{P}$ is not a k-stable partition.

Case 2: Let $|P_h| + |P_l| > k$, and suppose, without loss of generality, that $|P_h| > |P_l|$. If any $k - |P_h|$ individuals leave coalition P_l to join P_h, then k individuals would be better off while only $|P_l| - (k - |P_h|) = |P_h| + |P_l| - k < k$ would be worse off. That is, the partition $\mathcal{P}'$ generated from $\mathcal{P}$ by integrating coalitions P_h and P_l into one coalition of size k and another coalition of size $|P_h| + |P_l| - k$, is feasible and improves the payments of k individuals while reducing the payments of less than k. Therefore, $\mathcal{P}$ is not k-stable, since it is blocked by $\mathcal{P}'$. □

Corollary 2.1. *If q, $r \in \mathbf{N}$ are respectively the quotient and the reminder of the division of N*

by k, a k-stable partition (or, equivalently, a $\mathcal{P}_{k^*}$ partition) has q coalitions of size k and one of size r.

The above is a consequence of the previous proposition. Because there is at most one coalition of size less than k in a k-stable partition, the rest of the coalitions are of size k, with which, if $N = kq + r$, then the k-stable partition has q coalitions of size k and one of size r. Furthermore, this partition is stable in the usual sense as no individual can improve her payments without reducing the benefits of other individuals. Observe that k-stable partitions are constrained efficient because the largest possible number of individuals (kq) reach their maximum aggregate payoff $t(k)x_i$, and the rest obtain the maximum payoff $t(r)x_i$. The grand coalition is a special case of k-stable partitions when $k \geq N$.

Proposition 2.3. *Unlike the equal sharing rule, coalitions of a k-stable partition are not necessarily consecutive.*

Proof. This occurs because individuals do not have strict preference for forming coalition with more able individuals. Heterogeneity is not bad since individuals are indifferent between forming a group with other individuals with ability close or far from theirs. Given the sharing rule is proportional, this is evident because any individual $i \in \mathcal{N}$ obtains the same payoff in a coalition S than in any other coalition S' as long as both have the same cardinality. This is independent of the abilities of the rest of the members in each coalition (the composition of each group) and of the distribution of abilities. That is to say, $t(|S|)x_i = t(|S'|)x_i$ if and only if $|S| = |S'|$ as t is strictly increasing. □

Notice that there are $\binom{N}{k}\binom{N-k}{k}\dots\binom{k+r}{k}$ k-stable partitions, taking into account all the possible permutations of $\mathcal{N}$. Individual payoffs are equal for each partition consisting of q coalitions of size k and one of size $r < k$ as long as the individuals of this smallest coalition are the same, and this is independent of the composition of the other q coalitions. Indeed, let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions with the aforementioned characteristics such that the coalition of size

r forms. Then, each individual i that belongs to a coalition of size k under the partition $\mathcal{P}$ will be in another coalition of size k under partition $\mathcal{Q}$. Therefore, under both partitions, she will receive $t(k)x_i$, and the remaining r individuals will also preserve their payoffs.

Remark 2.1. *The partition consisting of q coalitions of size k and one of size r such that all the coalitions are consecutive, is also k-stable.*

Not every k-stable partition gives the same aggregate payments since they depend on the "position" of the coalition of size r. However, as long as $r = 0$ or the coalition of size r is the same in both partitions, they offer the same aggregate payoffs. Indeed, let $\mathcal{P}$ and $\mathcal{Q}$ be both k-stable partitions, such that the coalition of size r of both coincide, this is $P_r = Q_r$. Then,

$$t(k)\sum_{P_1} x_i + \ldots + t(k)\sum_{P_q} x_i + t(r)\sum_{P_r} x_i = t(k)\sum_{i=1}^{N-r} x_i + t(r)\sum_{S_r} x_i$$
$$= t(k)\sum_{Q_1} x_i + \ldots + t(k)\sum_{Q_q} x_i + t(r)\sum_{Q_r} x_i$$

since $\cup_{i=1}^{q} P_i = \cup_{i=1}^{q} Q_i$ and $P_r = Q_r$.

As we have shown earlier, with no exogenous restrictions of size of admissible coalitions (in other words, when $k \geq N$), the equilibrium partition is the one that consists of one large group which is also optimum since it maximizes both total and individual payoffs. Unrestricted size of coalitions may not be achievable in many situations since it can be costly in organizational or spatial terms. However, we shall show that, under certain conditions, we can still restrict the size of coalitions in such a way that the aggregate payoffs will be greater in the equilibrium under the proportional sharing rule than under equal sharing. We state this result in the following theorem and propose an algorithm to find an appropriate bound k.

Recall that under the equal sharing rule there is a unique equilibrium partition $\mathcal{S}$. Let S_M and S_m be the largest and smallest coalitions of $\mathcal{S}$, respectively. As Farrell & Scotchmer (1988) show, when the distribution of abilities is uniform, S_M consists of the most able individuals, while S_m, of the least able ones.

Theorem 2.1. *To obtain larger aggregate payoffs with the consecutive k-stable partition relative to $\mathcal{S}$, the stable partition under equal sharing, it is sufficient to fix a limit size $k \geq \hat{k}$, where $\hat{k}$ is given by as follows:*

1. *If N is divisible by $|S_M|$, then set $\hat{k} = |S_M|$;*
2. *If N is not divisible by $|S_M|$, then set $\hat{k}$ as the smallest integer that is larger than $|S_M|$ which satisfies one of the following conditions:*
   (i) *$\hat{k}$ is divisor of N, or*
   (ii) *the division of N by $\hat{k}$ gives a reminder r greater than $|S_m|$.*

Proof. We show that, in both cases, the consecutive k-stable partition under proportional sharing gives larger aggregate payoffs than the equilibrium under the equal sharing rule.

CASE 1: When $|S_M|$ is divisor of N and q is the quotient, the consecutive partition $\mathcal{P} = \{P_1, \dots, P_q\}$ where $|P_j| = |S_M|$ for each $j = 1, \dots, q$ is a $\hat{k}$-stable partition with $\hat{k} = |S_M|$. Under this partition, the total payoffs are

$$t(|S_M|)\sum_{P_1} x_i + \ldots + t(|S_M|)\sum_{P_q} x_i = t(|S_M|)\sum_{i=1}^{N} x_i.$$

On the other hand, the stable partition under equal sharing, $\mathcal{S} = \{S_m, \dots, S_M\}$, which is also consecutive, has aggregate payoffs of

$$t(|S_M|)\sum_{S_M} x_i + \ldots + t(|S_m|)\sum_{S_m} x_i < t(|S_M|)\sum_{S_M} x_i + \ldots + t(|S_M|)\sum_{S_m} x_i = t(|S_M|)\sum_{i=1}^{N} x_i.$$

Therefore, total payoffs of the k-stable partition are equal or higher.

CASE 2: If N is not divisible by $|S_M|$, then

(i) Let $\hat{k}$ be larger than $|S_M|$ and be a divisor of N. The proof of this case is analogous since

$$t(|S_M|)\sum_{S_M} x_i + \ldots + t(|S_m|)\sum_{S_m} x_i < t(\hat{k})\sum_{S_M} x_i + \ldots + t(\hat{k})\sum_{S_m} x_i = t(\hat{k})\sum_{i=1}^{N} x_i.$$

(ii) Let $\hat{k}$ be the smallest integer that is larger than $|S_M|$ which satisfies that the division of N by $\hat{k}$ gives a reminder r greater than $|S_m|$. Let q be the quotient of the division of N by $\hat{k}$, and let $\mathcal{P} = \{P_1, \ldots, P_q, P_r\}$ be the consecutive k-stable partition where $P_1, \ldots, P_q$ are the coalitions of size $\hat{k}$, and P_r the coalition of size r.

If there is h such that $\cup_{l=1}^{q} P_l = \cup_{l=1}^{h} S_l$ (clearly, $h \neq m$), by an analogous argument, the reduced partition $\mathcal{P}' = \{P_l\}_{l \neq r}$ gives higher total payoffs than the reduced partition $\{S_1, \ldots, S_h\}$ since the size of every P_l is at least as large as that of each S_l. Furthermore, $P_r = \cup_{l=h+1}^{m} S_l$ and then,

$$t(r)\sum_{P_r} x_i = t(r)\sum_{S_{h+1}} x_i + \ldots + t(r)\sum_{S_m} x_i > t(|S_{h+1}|)\sum_{S_{h+1}} x_i + \ldots + t(|S_m|)\sum_{S_m} x_i$$

since $r \geq |S_l|$ for $l = h+1, \ldots, m$. Therefore, total payoffs under $\mathcal{P}$ are higher than that under $\mathcal{S}$.

If there does not exist such an h, then let $\hat{h}$ be the minimum necessary such that $\cup_{l=1}^{q} P_l \subset \cup_{l=1}^{\hat{h}} S_l$. Since $r \geq |S_m|$, $\hat{h} \neq m$. Let $S_{\hat{h}_1}$, $S_{\hat{h}_2}$ be such that $S_{\hat{h}} = S_{\hat{h}_1} \cup S_{\hat{h}_2}$ and $\cup_{l=1}^{q} P_l = \cup_{l=M}^{\hat{h}-1} S_l \cup S_{\hat{h}_1}$. Then, since $\hat{k} \geq |S_l|$ for all $l = M, \ldots, \hat{h}-1, \hat{h}_1$, the aggregate payoffs are higher under the reduction $\{P_1, \ldots, P_q\}$ of $\mathcal{P}$ than under the reduction $\left\{S_M, \ldots, S_{\hat{h}-1}, S_{\hat{h}_1}\right\}$ of $\mathcal{S}$. For the remaining elements of both partitions we have something similar since $P_r = \cup_{l=\hat{h}+1}^{m} S_l \cup S_{\hat{h}_2}$ and $r \geq |S_l|$ for each $l = \hat{h}_2, \hat{h}+1, \ldots, m$.

Therefore, in each case, the aggregate payoffs are higher under the k-stable partition than under the equal sharing equilibrium. □

We have thus found a limit size k which is the least restrictive possible. The process to find the threshold value $\hat{k}$ can be illustrated by the following algorithm, represented by a pseudo code.

Result: Appropriate limit size of coalition k

Input: N, S_M, S_m;

Set k equal to $|S_M|$;

if *N is divisible by $|S_M|$* **then**

  return $\hat{k}$;

else

  while *the reminder of dividing N by $\hat{k}$ is less than $|S_m|$ or N is divisible by $\hat{k}$* **do**

    Set $\hat{k}$ equal to $\hat{k}+1$

  end

  return $\hat{k}$

end

Note that under a k-restricted size of coalition there may exist a different partition that gives larger total payoffs than our consecutive k-stable one; however, such partition will not form an equilibrium (it is not k-stable). Observe that the above theorem clearly implies that there are gains in efficiency under the proportional sharing rule compared with equal sharing even when we restrict the size of each coalition.

Chapter 3

Continuum of agents

3.1 The stable and efficient partitions

In this chapter, we consider a continuum of agents on $[0, 1]$. The abilities of the individuals are distributed on the interval $[0, M]$. To simplify calculations, we shall work with the uniform distribution, which allows us to re-scale the mass of individuals on the interval $[0, M]$, and to measure the size of a coalition as the measure of the subsets of $[0, M]$ (divided by M). Therefore, we can focus on partitions of the interval $[0, M]$ representing all the possible divisions of individuals into coalitions. Assume that the payoff to each individual that belongs to the coalition S is given by $t(\mu(S))x$, where $t : [0, M] \to \mathbf{R}^+$ is a strictly increasing function and μ is the Borel measure on $\mathbf{R}$. Therefore, the aggregate payoffs for a coalition S are given by $t(\mu(S)) \int_S x dF(x) = t(\mu(S)) \int_S x f(x) dx$ (taking the limit, with the Riemann integral).

Definition 3.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of measurable sets of the interval $[0, M]$, and let μ be the usual measure on $\mathbf{R}$. Define the binary relation $\sim$ as follows: $\mathcal{P} \sim \mathcal{Q}$ if there exists a bijection $\Phi : \mathcal{P} \to \mathcal{Q}$ such that $\Phi(P_l) = Q_h$ if and only if $\mu(P_l \bigtriangleup Q_h) = 0$, where $\bigtriangleup$ represents the symmetric difference between P_l and Q_h.*

Lemma 3.1. *The binary relation $\sim$ is an equivalence relation.*

Proof. It is immediate to verify that the binary relation $\sim$ is reflexive and symmetric. So, the only property we have to prove is transitivity. Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be measurable partitions of $[0, M]$ such that $\mathcal{P} \sim \mathcal{Q}$ and $\mathcal{Q} \sim \mathcal{R}$. Then, for each $P_l \in \mathcal{P}$, there exists $Q_h \in \mathcal{Q}$ and $R_s \in \mathcal{R}$ such that $\mu(P_l \vartriangle Q_h) = \mu(Q_h \vartriangle R_s) = 0$. Therefore, $\mu(P_l \vartriangle R_s) = 0$. Indeed,

$$\begin{aligned} P_l \vartriangle R_s &= (P_l \cap R_s^c) \cup (P_l^c \cap R_s) \\ &= (P_l \cap (Q_h \cup Q_h^c) \cap R_s^c) \cup (P_l^c \cap (Q_h \cup Q_h^c) \cap R_s) \\ &= [((P_l \cap Q_h^c) \cup (Q_h \cap P_l)) \cap R_s^c] \cup [((P_l^c \cap Q_h^c) \cup (Q_h \cap P_l^c)) \cap R_s] \\ &= [(P_l \cap Q_h^c \cap R_s^s) \cup (P_l \cap Q_h \cap R_s^c)] \cup [(P_l^c \cap Q_h^c \cap R_s) \cup (P_l^c \cap Q_h \cap R_s)] \\ &\subseteq (P_l \cap Q_h^c) \cup (Q_h \cap R_s^c) \cup (Q_h^c \cap R_s) \cup (P_l^c \cup Q_h) \\ &= (P_l \vartriangle Q_h) \cup (Q_h \vartriangle R_s). \end{aligned}$$

Therefore, $\mu(P_l \vartriangle R_s) \leq \mu(P_l \vartriangle Q_h) + \mu(Q_h \vartriangle R_s) = 0$, and hence, $\sim$ is an equivalence relation. □

Remark 3.1. Notice that $Q_h = \Phi(P_l)$ implies $\mu(P_l) = \mu(Q_h)$. Indeed, $P_l \subseteq P_l \cup Q_h = (P_l \vartriangle Q_h) \cup (P_l \cap Q_h)$, which implies $\mu(P_l) \leq \mu(P_l \vartriangle Q_h)) + \mu(P_l \cap Q_h) \leq \mu(Q_h)$. Analogously, $\mu(Q_h) \leq \mu(P_l)$, with which we have the result.

Denote by $[\mathcal{P}]$ the equivalence classes defined by the relation $\sim$. With the assumption of uniform distribution, we can consider the measure of a coalition as the measure of the set of the partition it represents. Also, we shall only consider partitions consisting of a finite number of coalitions in order to guarantee that all of them have relevant impacts on the equilibrium payoffs.

Lemma 3.2. *If two finite (or numerable) partitions $\mathcal{P}$ and $\mathcal{Q}$ belong to the same equivalence class, then each agent obtains the same payoffs in both the partitions. If there is a set of agents who obtains different payoffs, then this set has measure zero.*

Proof. Suppose on the contrary that there is $A \subseteq [0, M]$, such that $\mu(A) > 0$ and for each $x \in A$, $t(\mu(P_x))x > t(\mu(Q_x))x$. Notice that, since $\mathcal{P}$ is finite, there is $B \subseteq A$ of positive measure such that $B \subseteq P_l$ for some l. Thus, for each $x \in B$, $t(\mu(P_l))x > t(\mu(Q_x))x$, which implies for all $x \in B$, $Q_x \neq Q_h = \Phi(P_l)$. This is, for all $x \in B$, $x \in P_l \setminus Q_h \subseteq P_l \triangle Q_h$, which implies that $B \subseteq P_l \triangle Q_h$ y $\mu(B) \leq \mu(P_l \triangle Q_h) = 0$, which is a contradiction. □

Remark 3.2. If $\mathcal{P}$ and $\mathcal{Q}$ belong to the same equivalence class, then the payments for each coalition of positive measure are the same under both partitions. This follows from the previous remark since if two partitions are equivalent, the coalitions of positive measure in both partitions are of same measure.

We shall say that a partition $\mathcal{P} = \{P_l\}_{l \in L}$ of the interval $[0, M]$ is k-feasible if it is finite (i.e., it consists of a finite number of coalitions) and $\mu(P_l) \leq k$ for all $l \in L$. We shall call a partition k-stable if it is k-feasible and there is no other k-feasible partition such that the measure of the set of individuals that are better off is higher than the measure of the set of those that are worse off.

Remark 3.3. *Notice that if a partition is k-stable, then all the elements of its equivalence class are also k-stable. Thus, we can refer to classes of k-stable partitions.*

Lemma 3.3. *The coalition that maximizes the aggregate payoffs is the grand coalition.*

The argument is analogous to the finite case since we can distribute the integral on the interval $[0, M]$ into any partition. Let $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a partition of the interval $[0, M]$. Then, the aggregate payoffs under this partition are lower than those under the grand coalition:

$$t(\mu(P_1)) \int_{P_1} xf(x)dx + \ldots + t(\mu(P_l)) \int_{P_l} xf(x)dx < t(M) \int_0^M xf(x)dx.$$

Proposition 3.1. *With no restriction on the coalition size (equivalently, when $k \geq M$), the grand coalition is the unique stable and coalition under the proportional sharing rule.*

Proof. Note that, under the grand coalition every individual gets a higher payoff than under any other coalition structure since $t(\mu(P_i))x_i < t(M)x_i$. Therefore, no other partition can block the grand coalition. To show the uniqueness, let $\{P_1, \ldots, P_l\}$ be an arbitrary partition of $[0, M]$. Under this partition, each individual would be better-off if any two coalitions are merged into a single one. Moreover, no individual in the remaining coalitions would be worse off. Therefore, no coalition can improve upon the grand coalition. □

Proposition 3.2. *If we restrict attention to k-feasible coalitions, where $k \leq M$, then the equivalence classes of k-stable partitions will consist of those that have at most one coalition of measure less than k. Let M be equal to $kq + r$ where $q \in \mathbf{N}$ is the quotient and $r \in \mathbf{R}^+$ is the reminder of the division of M by k. Then, the classes of k-stable partitions will be the classes of $\mathcal{P}_{k^*}$ partitions—those that consist of q coalitions of size k and one of size r.*

Proof. To simplify notations, we shall use $|\cdot|$ instead of $\mu(\cdot)$. The proof of this result is analogous to the finite case. First, note that a $\mathcal{P}_{k^*}$ partition is k-stable since no coalition of size k can accept any other set of individuals of positive measure (and a deviation to a group of measure zero will have no effect). Moreover, any deviation of positive measure of agents from a partition of size k to the only partition of size less than k would at most lead to an improvement index of zero.

For the other implication, we proceed by contradiction. Suppose $\mathcal{P}$ is a k-stable partition that has two coalitions P_l and P_h such that $|P_l|, |P_h| < k$. We show that the payoffs for the individuals in both coalitions are higher when these two coalitions can merge into a single one. We have two possible cases:

Case 1: If $|P_l| + |P_h| \leq k$, each individual i in the coalition P_l obtains the payoff $t(|P_l|)x$ and each individual in the coalition P_h receives $t(|P_h|)x$. Clearly, both coalitions can be merged into one coalition $P_l \cup P_h$ without violating the feasibility condition. Moreover, each member i of this new coalition $P_l \cup P_h$ would receive payoff $t(|P_l| + |P_h|)x$, which is strictly higher than what she would obtain either in P_l or in P_h. Also, notice that by doing so, no individual will be worse off. That is, the partition $\mathcal{P}'$ obtained from $\mathcal{P}$ by merging coalitions P_l and P_h into one blocks $\mathcal{P}$. Therefore, $\mathcal{P}$ is not k-stable.

Case 2: Let $|P_l| + |P_h| > k$. Without loss of generality, assume that $|P_l| > |P_h|$. The individuals in coalition P_l would gain by accepting in their coalition a set of individuals of measure $k - |P_l|$ from coalition P_h and the new members of this coalition would also gain. This means, this union would make a set of individuals of size k better off since each member of the new coalition will receive $t(k)x$, while under the original partition they would obtain $t(|P_l|)x$ or $t(|P_h|)x$, respectively. On the other hand, there is a set of individuals that are worse off under the deeviation: those who where left out by the members that left coalition P_h to join coalition P_l; however, the measure of this set is $|P_h| - (k - |P_l|) = |P_h| + |P_l| - k < k$. Then, the partition $\mathcal{P}'$ generated from $\mathcal{P}$ by merging coalitions P_h and P_l into one of measure k and another coalition of size $|P_h| + |P_l| - k$ block $\mathcal{P}$ since it is feasible and improves the payoffs of a set of individuals of size k while reducing the payments of a set of size $|P_h| + |P_l| - k$. □

Remark 3.4. Using the same arguments we used for the finite case, it is easy to prove the following statements:

(i) $\mathcal{P}_{k^*}$ partitions are less efficient than the grand coalition.

(ii) Under the $\mathcal{P}_{k^*}$ partitions, the maximum possible measure of individuals (kq) get the maximum payments available: $t(k)x$.

(iii) Individuals do not prefer to form coalitions with the most able individuals, since they receive the same payments in any two coalitions as long as both have the same measure. That is to say, $t(\mu(S))x = t(\mu(S'))x \iff \mu(s) = \mu(S')$.

(iv) The $\mathcal{P}_{k^*}$ partitions are no necessarily consecutive, but the consecutive partition where the less able individuals form coalition of measure r is a $\mathcal{P}_{k^*}$ partition.

(v) Not every $\mathcal{P}_{k^*}$ partition has the same aggregated payments, only those where the coalition of size r matches.

The result of Theorem 1 can be generalized to the continuum case. To do this, consider a continuum of agents uniformly distributed on the interval $[0, M]$. Let S_M and S_m be the largest and smallest coalitions, respectively, in the unique stable partition, $\mathcal{S}$ under the equal sharing rule. Then, S_M consists on the most able individuals while S_m on the least able ones. The result goes as follows.

Theorem 3.1. *The consecutive k-stable partition is more efficient than the one under the equal sharing rule, it is sufficient to have a limit size $k \geq \hat{k}$, where $\hat{k}$ is determined as follows:*

1. *If the division of M by $\mu(S_M)$ has a integer quotient q, then set $\hat{k} = \mu(S_{max})$;*
2. *If not, set $\hat{k}$ as the smallest real number that is larger than $\mu(S_M)$, which satisfies one of the following conditions:*
   (i) *the quotient q of the division of M by $\hat{k}$ is an integer*
   (ii) *the integer division of M by $\hat{k}$ (integer quotient q) gives a reminder r greater than $\mu(S_m)$.*

Proof. The proof is analogous to the finite case. We show that, in both cases, the consecutive k-stable partition under proportional sharing rule yields larger aggregate payoffs than the equilibrium under equal sharing.

CASE 1: When the quotient q of the division of M by $|S_M|$ is an integer, the consecutive partition $\mathcal{P} = \{P_1, \ldots, P_q\}$ where $|P_l| = |S_M|$ for each $l = 1, \ldots, q$ is a k-stable partition for $k = |S_M|$. Under this partition, the total payoffs are

$$t(|S_M|) \int_{P_1} xf(x)dx + \ldots + t(|S_M|) \int_{P_q} xf(x)dx = t(|S_M|) \int_0^M xf(x)dx$$

On the other hand, they aggregate payoffs associated with the equilibrium partition under equal sharing, which is also consecutive, are given by:

$$t(|S_M|) \int_{S_M} xf(x)dx + \ldots + t(|S_m|) \int_{S_m} xf(x)dx < t(|S_M|) \int_{S_M} xf(x)dx + \ldots + t(|S_M|) \int_{S_m} xf(x)dx$$
$$= t(|S_M|) \int_0^M xf(x)dx$$

Therefore, aggregate payoffs of the k-stable partition are higher.

CASE 2: If the division of M by $|S_M|$ is not an integer,

i) Let k be the smallest real number larger than $|S_M|$ where the division of M by k has an integer quotient q. It simply means that there exists a $q \in \mathcal{N}$ such that $M = kq$. The proof of this case is analogous, since

$$t(|S_M|) \int_{S_M} xf(x)dx + \ldots + t(|S_m|) \int_{S_m} xf(x)dx < t(\hat{k}) \int_{S_M} xf(x)dx + \ldots + t(\hat{k}) \int_{S_m} xf(x)dx$$
$$= t(\hat{k}) \int_0^M xf(x)dx.$$

ii) If there is an smallest $k \in \mathbf{R}$ such that $M = kq + r$, where q in an integer and $r \geq |S_m|$, then we can consider the two following cases.

If there exists h such that $\cup_{l=1}^{q} P_l = \cup_{l=M}^{h} S_l$ (clearly $h \neq m$), by an argument analogous to the furthers, the reduced partition $\mathcal{P}' = \{P_l\}_{l \neq r}$ gives higher total payments than the reduced partition $\{S_M, \ldots, S_h\}$ since the size of every P_l is at least as large as S_l. Furthermore, $P_r = \cup_{l=h+1}^{m} S_l$ and then,

$$t(r) \int_{P_r} xf(x)dx = t(r) \int_{S_{h+1}} xf(x)dx + \ldots + t(r) \int_{S_m} xf(x)dx$$
$$> t(|S_{h+1}|) \int_{S_{h+1}} xf(x)dx + \ldots + t(|S_m|) \int_{S_m} xf(x)dx$$

since $r \geq |S_l|$ for $l = h+1, \ldots, m$. Therefore, total payments under $\mathcal{P}$ are higher than under $\mathcal{S}$.

If there does not exist such h, let $\hat{h}$ be the minimum necessary such that $\cup_{l=1}^{q} P_l \subset \cup_{l=M}^{\hat{h}} S_l$. Since $r \geq |S_m|$, $\hat{h} \neq m$. Let $S_{\hat{h}_1}, S_{\hat{h}_2}$ such that $S_{\hat{h}} = S_{\hat{h}_1} \cup S_{\hat{h}_2}$ and $\cup_{l=1}^{q} P_l = \cup_{l=M}^{\hat{h}-1} S_i \cup S_{\hat{h}_1}$. Then, since $k \geq |S_l|$ for all $l = M, \ldots, \hat{h}-1, \hat{h}_1$, the payments are higher under the reduction $\{P_1, \ldots, P_q\}$ of $\mathcal{P}$ than under the reduction $\left\{S_M, \ldots, S_{\hat{h}-1}, S_{\hat{h}_1}\right\}$ of $\mathcal{S}$. For the remaining elements of both partitions occur something similar, since $P_r = \cup_{l=\hat{h}+1}^{m} S_l \cup S_{\hat{h}_2}$ and $r \geq |S_l|$ for each $l = \hat{h}+1, \ldots, m, \hat{h}_2$.

Therefore, in each case, the payments are higher under the k-stable partition than under the equal sharing equilibrium. □

As in the finite case, this result gives us an idea of how restrictive we could be with the size of coalitions and still achieve gains in efficiency. To this end, we have argued that it may not be feasible to form the grand coalition when the number of individuals is too large, and we have proposed an exogenous restriction on size to analyze how far one can push to maintain more efficiency under the proportional rule relative to the equal sharing rule. The next step is to analyze how to obtain such restrictions endogenously. We shall show that such size restrictions may emerge due to the presence of some organizational costs. In particular, we shall introduce individualized cost functions that depend on group size, which would put limits on forming very large coalitions.

3.2 Expansion costs and endogenous size restriction

We now consider a situation where each agent faces a cost of forming coalition, i.e., the larger the coalition she belongs to, the higher is the cost she incurs. Such costs may be thought of as the disutility that an individual faces by working in large groups. Let $c_i(|S|)$ be the cost function of the agent i with ability x_i, which is strictly increasing and convex. Given two individuals, $i, j \in \mathcal{N}$, we say i is "more averse to large groups" than j if $c_i(|S|) > c_j(|S|)$. Given the type-dependent cost functions $c_1(\cdot)$, ..., $c_N(\cdot)$, the aggregate payoffs of a given coalition S is given by:

$$t(|S|)\int_S xf(x)dx - C(|S|) = t(|S|)\int_S xf(x)dx - \int_S c_i(|S|)di.$$

An individual i with ability x_i solves

$$S^*(x_i) = \underset{S}{\text{argmax}} \left\{ u_i(|S|) \equiv t(|S|)x_i - c_i(|S|) \right\}.$$

Because $u_i(\cdot)$ depends only on the size of the group individual i belongs to, she would simply maximize $u_i(|S|)$ with respect to S. Notice that $u_i(|S|)$ is strictly concave. The first order condition of i's maximization problem is given by:

$$t'(|S|)x_i = c_i'(|S|) \quad \Longleftrightarrow \quad x_i = \frac{c_i'(|S|)}{t'(|S|)} \equiv f_i(|S|),$$

where $f_i(\cdot)$ is an increasing and positive function. The individual optimum is depicted in Figure 3.1.

Figure 3.1: The optimal choice of coalition of agent i

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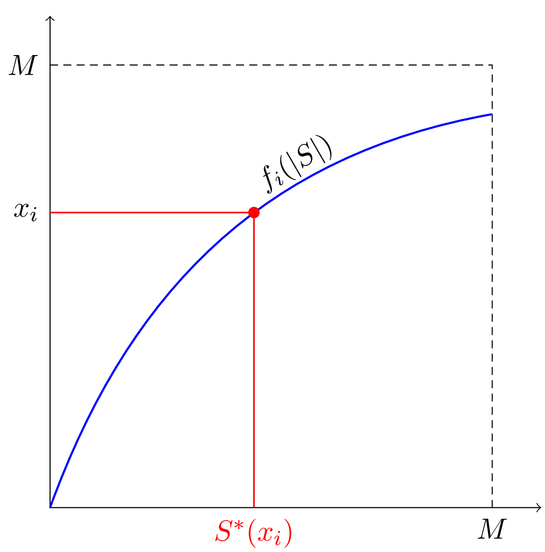


Source: Own elaboration

Note that given her level of "aversion to large groups", an individual with higher ability would be more "tolerant" to heterogeneity within her group (in other words, to work with less able individuals) since the positive effect of t is larger. That can be enunciated in the following lemma.

Lemma 3.4. *The optimum group size is monotonically increasing in ability.*

This follows from the fact that $f_i(|S|)$ is an increasing function, whereas x_i is a constant function. When an individual is less averse to big groups, $c_i'(|S|)$ will increase slower, which will conduce to a function f_i with lower slope. On the other hand, an individual with higher costs of expansion c_i will have a f_i of higher slope. Clearly, for any i if it is the case that $f_i(M) < M$, then $S^*(x_i)$ must be the grand coalition when $x_i \geq f_i(M)$ (because $x_i = f_i(|S|)$ yields $|S| > M$). So, in general, and individual would not optimally choose the grand coalition, which means a loss of efficiency under the proportional sharing rule. Moreover, with distinct individual optima it is difficult to characterize a stable and efficient partition.

However, it is possible to construct examples where $S^*(x_i) = S^*$ for all $i \in \mathcal{N}$, i.e., all agents have the same optima. In this case, letting $k = |S^*|$, the equilibrium is k-stable. Let the individual cost function be

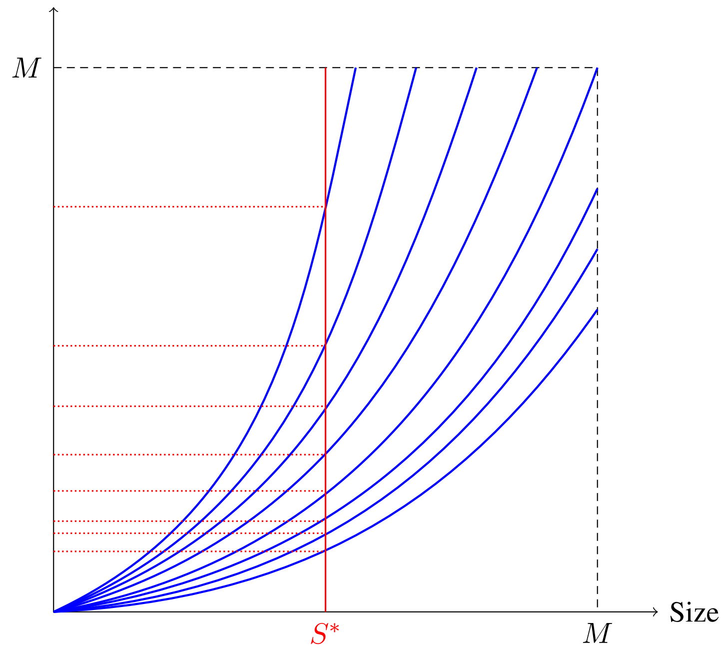
$$c_i(|S|) = x_i\, C(|S|) \quad \text{for all } \; i \in \mathcal{N}.$$

The function $C(|S|)$ is an aggregate cost component that depends on the group size, and hence, $c_i(|S|)$ is simply a proportional cost-sharing rule. In this case, S^* solves

$$x_i = f_i(|S|) = \frac{x_i C'(|S|)}{t'(|S|)} \quad \Longleftrightarrow \quad C'(|S|) = t'(|S|).$$

Note that this situation is equivalent to the proportional surplus sharing rule because $\hat{t}(|S|) \equiv t(|S|) - C(|S|)$ represents the net surplus of a coalition S where $u_i(|S|) = \hat{t}(|S|)x_i$. This situation is depicted in Figure 3.2. The above discussion is summarized in the following proposition.

Figure 3.2: The $|S^*|$-stable partition.



Source: Own elaboration

Proposition 3.3. *Let the individual cost function implies proportional cost sharing, i.e.,* $c_i(|S|) = x_i\bar{C}(|S|)$. *Then, the unique consecutive constrained efficient stable partition involves* q *coalitions each of size* $|S^*|$ *and one coalition of size* $r < |S^*|$.

Chapter 4

Quasi-proportional rules

In many organizations, the division of surplus does not obey neither equal sharing nor proportional sharing. Rather, surplus is shared according to some non-linear sharing rule. For example, a fraction of an organization's total profits is divided equally among its members, and the remaining fraction paid to each one according to her productivity or ability (as "bonus"). Thus, in this chapter, we consider a sharing rule that is a convex combination of the equal and proportional sharing rules, which we term the *quasi-proportional* sharing rule. The aggregate surplus of a coalition S is given by:[2]

$$t(|S|) \int_S x f(x) dx.$$

Under the quasi-proportional rule, which divides $1 - \alpha_S$ fraction of the surplus of a given coalition S equally, and the remaining fraction, proportionally among the members of S, an individual $i \in S$ consumes

$$\begin{aligned} u_i(|S|) &= (1 - \alpha_S) \cdot \frac{t(|S|)}{|S|} \int_S x dx + \alpha_S \cdot \frac{x_i}{\int_S x dx} t(|S|) \int_S x dx \\ &= (1 - \alpha_S) \cdot \frac{t(|S|)}{|S|} \int_S x dx + \alpha_S \cdot t(|S|) x_i. \end{aligned}$$

[2] If the abilities are uniformly distributed on $[0, M]$, then the aggregate surplus becomes $\frac{1}{M} \int_S x dx$, which can be written as $\int_S x dx$ if we re-scale the support $[0, M]$ to $[0, 1]$.

The above sharing rule is indeed a convex combination of the equal and proportional sharing rules, with corresponding weights $1-\alpha_S$ and α_S, respectively. In the following proposition, we analyze the effect of changes in α_S and ability, x_i.

Proposition 4.1. *Let $\lambda_S = \frac{1}{|S|}\int_S xdx$ be the average ability of a given group S, $\varepsilon(|S|) \equiv \frac{t'(|S|)|S|}{t(|S|)}$ be the elasticity of $t(\cdot)$ with respect to group size, and $|S_i^*| \equiv |S^*(x_i,\, \lambda_S,\, \alpha_S)|$ be the optimal coalition size of individual i. If $\varepsilon(|S|)$ is decreasing (increasing) in group size, S, then the optimal choice of individual i, $|S_i^*|$ is increasing in both α_S and the the ability of individual i relative to S, x_i/λ_S.*

Proof. We have assumed that, in each coalition S, the minimum ability member has ability 0, which allows us to think of coalition S as the interval $[0, |S|]$.[3] The first order condition of individual i's payoff maximization problem is given by:

$$\begin{aligned} &(1-\alpha_S)\frac{t'(|S|)|S| - t(|S|)}{|S|^2}\int_S xdx + \alpha_S t'(|S|)x_i = 0 \\ \iff\ & 1 + \left(\frac{\alpha_S}{1-\alpha_S}\right)\frac{x_i}{\lambda_S} = \frac{1}{\varepsilon(|S|)}\,. \end{aligned} \tag{4.1}$$

Now an increase in α_S, i.e., greater weight on the proportional rule, or an increase in x_i/λ_S increases the left-hand-side of the above equation. Therefore, optimality dictates that $\varepsilon(|S|)$ must decrease. Therefore, if $\varepsilon(|S|)$ is decreasing (increasing) in $|S|$, then S_i^* would be increasing (decreasing) in both α_S and x_i/λ_S. □

Clearly, when $\alpha_S = 0$ for all S, we have the model of Farrell & Scotchmer (1988). In this case, under uniform distribution, there is a unique stable structure in which coalitions are consecutive, and more able individuals form larger coalitions. On the other hand, at $\alpha_S = 1$ for all S, we have our model with proportional sharing rule where the grand coalition is the only stable and efficient partition. What happens with stability and efficiency for $\alpha_S \in (0, 1)$ for any S is difficult to determine, and left as a topic for future research.

[3] The analysis trivially generalizes to the case when a generic coalition is of the form $[l, |S| + l]$ for $l > 0$.

Chapter 5

Conclusion

When heterogeneous individuals form groups in order to carry out productive activities, it may be intuitive to believe that the easiest and fairest way to share their group surplus is to distribute it equally. However, since individuals are not equal, equal sharing of surplus implies to loss of efficiency due the trade-off between size or homogeneity each coalition faces. Such loss in efficiency can be mitigated by deviating from equal sharing, e.g., the proportional sharing rule.

We show that, under proportional sharing rule, there is a unique stable and efficient coalition structure, which is the grand coalition. We then find conditions under which smaller coalitions can form a (constrained) efficient and stable coalition structure. We show that such exogenous bounds on coalition size can be endogenized by introducing individualized expansion costs. When such costs implies proportional cost sharing, there is an efficient and stable structure. Finally, we consider a convex combination of the equal and proportional sharing rules, under which individual optimum size may vary non-monotonically with respect to ability.

Many questions are left open to answer. We have discussed situations where efficient coalition structure differs from the individual optimum. Moreover, sometimes the individual optimum is monotone in ability, and it is non-monotonic in other situations. The analyses of efficient and stable coalition structures are on the agenda for future research.

References

Banerjee, S., Konishi, H., & Sönmez, T. (2001). Core in a simple coalition formation game. *Social Choice and Welfare*, *18*, 135-153.

Bogomolnaia, A., & Jackson, M. (2002). The stability of hedonic coalition structures. *Games and Economic Behavior*, *38*, 201-230.

Farrell, J., & Scotchmer, S. (1988). Partnerships. *The Quarterly Journal of Economics*, *103*, 279-297.

Kaneko, M., & Wooders, M. (1982). Cores of partitioning games. *Mathematical Social Sciences*, *3*, 313-327.

Kaneko, M., & Wooders, M. (1986). The core of a game with a continuum of players and finite coalitions: The model and some results. *Mathematical social sciences*, *12*, 105-137.

Pycia, M. (2012). Stability and preference alignment in matching and coalition formation. *Econometrica*, *38*, 201-230.