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NEGOTIATION WHEN THE SIZE OF THE PIE DEPENDS ON HOW IT IS CUT

Introduction

Rubinstein's (1982) infinitely repeated ultimatum game between two players provides a paradigm to analyze negotiation. We extend his analysis to the case in which the benefits to be distributed depend on the result of the negotiation, that is, on whose proposal is finally accepted and on how the benefits are distributed. Each player draws her proposals from a different set of possibilities, which is not assumed to be convex. We obtain general theorems classifying and calculating the results of this kind of negotiation game. One implication is a theoretical characterization of when a negotiation game becomes a principal agent game. Most of the paper is devoted to the proof of these results. We also give two brief examples of their application.

The first example is one of the problem which motivated our research. We analyze the simple case of negotiation (sharing a fixed pie) when the players have a concept of fairness. In this case, their final utility depends not only on their share of the pie but also on how fairly they feel it was cut; unfairness causing a loss of utility which may be different for each player. We find that the rule of sharing in half holds for a range of players including morally deficient players and players with weak characters (that is, not very willing to punish the other for being unfair).

The second example is Cournot duopoly. We consider a market supplied by two firms which are deciding how much to produce. Their proposals consist of taking observable actions in preparation for production, which announce their intended output. The other firm accepts the proposal if it decides to maximize its profits taking as given the other firm's output as implied by its actions. Instead, it may opt for a different level of production, and act accordingly. In such a situation, which is modeled by the infinitely repeated ultimatum game, not all Cournot equilibria (at which each firm has no incentives to deviate from its production level given the other firm's production level) are equilibria in the negotiation game.

The plan of the paper is as follows. In section II we describe the general negotiation game which we solve, and write down the main results. In section III we apply these results to the simple case of negotiation when the players have a concept of fairness. In section IV we give our examples on duopoly. Section V constructs the proof of the general negotiation game. Then follow the final remarks and references. The Appendix contains the details of the proofs of the preceding sections.

The main results

Joan and Mary play a negotiation game G^{∞} á la Rubinstein, consisting of an infinite alternated succession of two ultimatum games, G^J and G^M (see Figure 1). In G^J Joan makes a proposal which has utility payoffs (u, v) (for Joan and Mary respectively) chosen from her possibility set K^J or opts for the reserve utility payoffs, $u_R \geq 0$ for Joan and $v_R \geq 0$ for Mary. Mary decides either to accept the proposal, to reject it to make her own proposal, or to opt for the reserve utility. In the game G^M the roles are reversed, and Mary makes a proposal (u, v) from her possibility set K^M. The proposals (u, v) are non-negative and the sets K^J, K^M are compact (and not necessarily connected). Joan and Mary discount future utility with discount rates ρ, μ, respectively. In the case of two payoffs with value zero we shall still suppose that the one occurring first is preferred. In the game G^∞ Joan is the first player to make a proposal. We also consider the game $G^{\infty\prime}$ in which Mary plays first, and a game in which the players decide who should make the first offer.

To solve these games, we define the functions $f : D^M \to \mathbf{R}$, $g : D^J \to \mathbf{R}$, where

$$D^M = [-\infty, v^+],\ \ v^+ = \sup(K^J(v)),\ \ K^J(v) = \{v : \exists\ (u, v) \in K^J\}$$

$$D^J = [-\infty, u^+],\ \ u^+ = \sup(K^M(u)),\ \ K^M(u) = \{u \mid \exists\ v : (u, v) \in K^M\}$$

by the formulae:

$$f(x) = \max\{u \mid \exists\ v \geq x : (u, v) \in K^J\},$$

$$g(x) = \max\{v \mid \exists\ u \geq x : (u, v) \in K^M\}.$$

f and g are monotonically decreasing functions which are continuous on the left because K^J and K^M are compact. These functions represent the maximum utility that each player can obtain by using her own set of proposals under the constraint of offering the other player a minimum utility x.

We shall now state the payoffs of the negotiation games in the case when we set $\rho = e^{-\alpha\tau}$, $\mu = e^{-\beta\tau}$ and let τ tend to zero. To do so we need the following definitions.

Definition 1 A combination of strategies supports a *perfect limit equilibrium* with payoffs (u_0, v_0) if and only if $\forall\ \varepsilon > 0\ \exists\ \tau_0 > 0 : \forall\ \tau < \tau_0$ there exists a strategic combination which supports a perfect equilibrium with payoffs (u_τ, v_τ), and $|u_\tau - u_0| < \varepsilon$, $|v_\tau - v_0| < \varepsilon$. The *limit payoffs* are the payoffs of the perfect limit equilibria.♦

Definition 2 Let PC^1 be the set of piece-wise once differentiable functions with at most a finite number of discontinuities whose derivatives are piece-wise continuous except possibly at a finite number of points, with left and right limits existing everywhere.♦

Definition 3 Define the sets

$$\Sigma M_0 = \{v \in [v_R, v^+] \cap K^J(v) \mid f(v) \geq u_R \text{ and } f(v) \leq u^+ \Rightarrow g(f(v)) < v\},$$

$$EM_0 = \{v \in [v_R, v^+] \cap K^J(v) \mid f(v) \geq u_R,\ f(v) \leq u^+ \text{ and } g(f(v)) = v\},$$

$$FM_0 = \{v \in [v_R, v^+] \cap K^J(v) \mid f(v) \geq u_R,\ f(v) \leq u^+ \text{ and } g(f(v)) > v\},$$

$$\Gamma M_0 = \{v \in \Sigma M_0 \cup EM_0 \mid \exists\ \varepsilon > 0 : [\varepsilon^2 v, v) \cap (\Sigma M_0 \cup EM_0) = \varnothing\};$$

and write $f^\varepsilon(v) = -f'(v)v/f(v)$ for the negative elasticity of f where it exists. Define also the following conditions for points $v_0 \in [v_R, v^+]$.

(1) $v_0 \in [v_R, v^+]$ satisfies LHC1 (left hand condition 1) if $\exists\, \varepsilon > 0$ for which, writing $I^- = (v_0-\varepsilon, v_0)$, $I^- \subseteq E^M{}_0$ and on I^- f has continuous derivatives, has g as its inverse, and $f^\varepsilon(v) \leq \alpha/\beta$ on I^-, without equality occuring on subintervals of I^-.
(2) $v_0 \in [v_R, v^+]$ satisfies RHC1 (right hand condition 1) if $\exists\, \varepsilon > 0$ for which, writing $I^+ = (v_0, v_0+\varepsilon)$, $I^+ \subseteq E^M{}_0$ and on I^+ f has continuous derivatives, has g as its inverse, and either $f^\varepsilon(v) > \alpha/\beta$ arbitrarily closely to v_0 on I^+ or $f^\varepsilon(v) = \alpha/\beta$ on subintervals of I^+ arbitrarily closely to v_0.
(3) $v_0 \in [v_R, v^+]$ satisfies LHC2 if $\exists\, \varepsilon > 0$ for which $\Sigma^M{}_0 \cap (v_0-\varepsilon, v_0) = \varnothing$.
(4) $v_0 \in [v_R, v^+]$ satisfies RHC2 if $\forall\, \varepsilon > 0$ $\Sigma^M{}_0 \cap (v_0, v_0+\varepsilon) \neq \varnothing$.♦

We shall simplify the passage to the limit as τ tends to zero by considering the case in which f and g are PC^1, and that the number of boundary points of the sets $\Sigma^M{}_0$, $E^M{}_0$, $F^M{}_0$, $\Gamma^M{}_0$ defined in section II and $\Gamma^M{}_\tau$ have a common bound N to the number of their boundary points. This excludes only pathological functions which have a lot of variation about some points but which are not particularly relevant economically. Further, we shall need the limits of the sets $\Phi^M{}_\tau$, $\Phi^J{}_\tau$ defined above (see Propositions 2 and 4). We find that Joan's optimal proposals (U^{J*}, V^{J*}), are constructed as follows if they exist. Let

$$\underline{E}^M{}_0 = \{v \in E^M{}_0 : \text{v satisfies one left hand and one right hand condition}\}.$$

For each $v^* \in \Gamma^M{}_0 \cup \underline{E}^M{}_0$ let

$$U^{J*} = f(v^*),\ V^{J*} \in \Phi^M{}_0.$$

$\Phi^M{}_0$, the limit set of proposals which Joan has available to offer Mary at least v^* and to obtain U^{J*}, is defined in Definition 7 and some of its properties are stated in Lemma 2. $\Phi^M{}_0$ is contained and often equal to the set $\{v \in [v^*, v^+] \mid (U^{J*}, v) \in K^J\}$, which is usually a singleton.

Mary's optimal proposal (U^{M*}, V^{M*}) are constructed as follows if they exists. Let

$$\Gamma^J{}_0' = \{u \in \Gamma^J{}_0 \mid \mu g(u/\rho) \notin \underline{E}^M{}_0\}.$$

For each u^*, let

$$V^{M*} = g(u^*),\ U^{M*} \in \Phi^J{}_0 \subseteq \{u \in [u^*, u^+] \mid (u, V^{M*}) \in K^M\}.$$

Similarly we define $\underline{E}^J{}_0$, $\Gamma^M{}_0'$ and construct the limit proposals of $G^{\infty\prime}$.

Theorem 1 Suppose that f and g are PC^1 and that the sets $\Sigma^M{}_0$, $E^M{}_0$, $F^M{}_0$, $\Gamma^M{}_0$ defined in section II and $\Gamma^M{}_\tau$ have a common bound N to the number of their boundary points.
(1) There is a one to one correspondence between the set $\Gamma^M{}_0 \cup \underline{E}^M{}_0$ and the subgame perfect limit equilibria payoffs of G^∞ in which the accepted proposal is Joan's and between the set $\Gamma^J{}_0'$ and the limit equilibria payoffs in which the accepted proposal is Mary's. $\underline{E}^M{}_0 = \underline{E}^J{}_0$ so these limit equilibria payoffs are the same except that they are proposed by Mary instead of Joan. Only these equilibria have the property that the accepting player can propose the payoffs herself. In the

other cases the players will agree that the player making the equilibrium proposal should begin the game. If both sets are empty each of the games G^{∞} and $G^{\infty\prime}$ is worth (u_R, v_R).

(2) If $\Gamma^M{}_0 \cup E^M{}_0$ is a singleton and $\Gamma^J{}_0{}'$ is empty, the game is equivalent to a principal agent game in which Joan proposes; both players agree she should start the game and she obtains a payoff $f(v_R)$. The symmetric statement holds for Mary.♦

We give some examples of the application of these results in sections III and IV, before proving them in section V.

Negotiation when there is a value of equality

We suppose that in the infinitely repeated ultimatum game, both players have the moral belief that the resulting distribution should be equal. To represent such players we need a description of normative behavior. García-Barrios and Mayer (1996) introduce the concept of a postconventional agent whose normative behavior depends on the context and may be influenced by payoffs, and thus may have various degrees of moral strength. Their theory is based on Tapp, Gunnar & Keating's (1983) characterization of the development stages of normative reasoning and on the theories of psychological congruence (Heider, 1958, Festinger, 1957, 1964). They use these concept to explain the behavior observed in experimental ultimatum games and predict some additional behaviors.[1] Here we apply their model of agency to the infinitely repeated version of the full ultimatum game, which includes the possibility of the responder punishing the proposer at a certain cost.

We suppose that Joan plays with Mary a repeated ultimatum game with punishment (Figure 3). When Joan makes a proposal, she offers Mary a share s^M which implies she remains with a share s^J = $1 - s^M$. Joan knows, however, that if Mary accepts she may impose a punishment for unfair proposals which consists of removing δx units from Joan's share at a cost of x units of her own share. Joan's proposal set is therefore

$$K^J = \{(s^J - \delta x - D^J(s^M - x - s^J + \delta x), s^M - x - D^M(s^J - \delta x - s^M + x)) : 0 \leq s^J \leq 1\}$$

while Mary's is

$$K^M = \{(s^J - x - D^J(s^M - \delta x - s^J + x), s^M - \delta x - D^M(s^J - x - s^M + \delta x)) : 0 \leq s^J \leq 1\}$$

where

[1] In the case of the single-shot ultimatum game, if the players only care for the monetary payoffs, game-theoretical analysis predicts a subgame perfect equilibrium in which the responder will accept any share offered by the proposer, who will offer the smallest unit of currency available. Furthermore, the responder's threats to reject any other offer will not be credible. However, the following conducts have robust empirical support (see Kahneman, Knetsch and Thaler, 1986 and Camerer & Thaler, 1995):

(a) Usually, individuals playing the role of proposer will offer a larger share than predicted.

(b) Individuals playing the role of responder will usually be reluctant to accept positive offers which imply a very unequal and therefore unfair distribution.

(c) Responders will be willing to pay a cost for punishing unfair offers.

(d) Proposers may take advantage of information asymmetries to increase their share (appearance of fairness is enough).

$$D^P(T) = d_0^P(T) \text{ if } T \geq 0,\ D^P(T) = d_1^P(-T) \text{ if } T \leq 0,\ P \in \{J, M\},$$

and

$$d_i^P,\ d_i^{P\prime},\ d_i^{P\prime\prime} > 0,\ i = 0, 1.$$

$D^J(T)$ and $D^M(T)$ represent the cognitive dissonance that each player feels when the outcome is unfair, where T is how much more the other player's payoff was then the player's own, after punishment. If $T > 0$ the dissonance represents the emotional discomfort arising from the other player having obtained a larger share, while if $T < 0$ it represents the discomfort arising from having obtained a larger share than the other player. These need not be the same.

Theorem 2 Proposing a share so unfair that the other player will be provoked to punish will be counterproductive. Each player has a maximum level of unfairness she will tolerate without punishing. Mary's is: $T_\delta^M = 0$ if $d_1^{M\prime}(0) \geq (\delta - 1)^{-1}$; otherwise $T_\delta^M = \min\{T, 1\}$, where T solves $d_1^{M\prime}(T) = (\delta - 1)^{-1}$ (the maximum level of unfairness is 1; Joan's maximum tolerance for unfairness is similarly defined by changing M for J). If $T_\delta^M = 0$ we say Mary has a strong character, while if $T_\delta^M > 0$ we say she has a weak character. Each player may have a maximum level of unfairness she is willing to impose. Joan's is: $s_+^J = \frac{1}{2}$ if $d_0^{J\prime}(0) \geq 1$; otherwise $s_+^J = \min\{s, 1\}$ where s solves $d_0^{J\prime}(s) = 1$ (for Mary's change J to M). If $s_+^J = \frac{1}{2}$ we say Joan is morally strong while if $s_+^J > \frac{1}{2}$ we say she is morally deficient. Thus the maximum shares that Joan and Mary can propose for themselves with advantage is:

$$s_{Max}^J = \min\{\frac{1 + T_\delta^M}{2}, s_+^J\},\ s_{Max}^M = \min\{\frac{1 + T_\delta^J}{2}, s_+^M\},$$

and correspondingly the minimum share they will obtain in proposals from the other player is:

$$s_{Min}^J = 1 - s_{Max}^M,\ s_{Min}^M = 1 - s_{Max}^J.$$

There is a unique solution to the negotiation problem, which occurs between these extreme feasible values (see Figure 4 for a graphic example). Since the proposals do not provoke punishment, the player's proposal sets reduce to the same proposal sets, and f and g are inverses in the interval of feasible values. The negative elasticity of f is a decreasing function of Joan's share s^J. Let

$$\varepsilon_1 = \frac{(1 - 2d_0^M(0))(1 - 2d_1^{J\prime}(0)}{(1 + 2d_1^J(0))(1 + 2d_0^{M\prime}(0)},\ \varepsilon_2 = \frac{(1 + 2d_1^M(0))(1 + 2d_0^{J\prime}(0)}{(1 - 2d_0^J(0))(1 - 2d_1^{M\prime}(0)}).$$

The solution occurs at $s^J = \frac{1}{2}$ if

$$\alpha/\beta \in (\varepsilon_1, \varepsilon_2);$$

it occurs on $[s_{Min}^J, \frac{1}{2})$ for smaller and on $(\frac{1}{2}, s_{Max}^J]$ for larger values of α/β. The extreme point s_{Max}^J is not reached if Joan's limit to a large, unfair share is self-imposed, and conversely for the point s_{Min}^J. ♦

The main result of this theorem is that it constructs the rule of sharing by half. It is clear that players who are morally strong or have strong characters will share by half. In these cases the feasible proposals reduce to the point $(\frac{1}{2}, \frac{1}{2})$. But the theorem implies that for a range of comparative impatience α/β, even players who are morally deficient and have weak characters will share by half. This will happen because treating the other unfairly involves dissonance for both players. One feels bad because she tries to get more than her fair share (but not necessarily bad enough not to do so if she were a principal agent) and the other feels bad because she is being cheated (but not necessarily bad enough to pay for punishing the other). These feelings on their own or combined imply that when the players negotiate they stick to the proposal $(\frac{1}{2}, \frac{1}{2})$ unless one player is that much more impatient than the other.

Duopoly

We consider some examples in which a market is supplied by two firms which are deciding how much to produce. Together the firms face a given demand $q = D(p)$, where q is the total quantity demanded and p the price. The firms have production costs (including capital costs) given by functions $C_i(q_i)$, where q_i is the quantity produced by Firm i, i = 1, 2. To analyze a situation analogous to Rubinstein's negotiation game, but not necessarily consisting of actual negotiations, we suppose that the firms are in a planning stage at which their actions fully announce their intended output. If Firm 1 plans to produce q_1, Firm 2 takes this as a proposal and signal acceptance by taking actions to produce a quantity q_2 which maximizes its benefits when taking Firm 1's production as given. Let

$$\Pi_1(q_1, q_2) = pq_1 - C_1(q_1),\ \Pi_2(q_1, q_2) = pq_2 - C_2(q_2), \text{ where } p = D^{-1}(q_1+q_2).$$

be the profits of each firm when the price is p. Let

$$q_2^*(q_1) = \text{argmax}\{\Pi_2(q_1, q_2) \text{ subject to } p = D^{-1}(q_1+q_2)\}.$$

Then when Firm 1 plans to produce q_1, Firm 2 accepts if it produces

$$Q_2(q_1) = \begin{array}{ll} q_2^*(q_1) & \text{if } \Pi_2(q_1, q_2^*(q_1)) \geq 0, \\ 0 & \text{if } \Pi_2(q_1, q_2^*(q_1)) < 0. \end{array}$$

If instead of accepting Firm 1's proposal Firm 2 takes actions to produce a different amount q_2, Firm 1 may reconsider its plans and take different actions accordingly. We assume that while the firms have not reached an agreement (so one firm has accepted the other's proposal), the delays cause a discount which covers the costs of the actions taken and the delay in the profits of the enterprise. In the language of our negotiation game, Firm 1's proposal set is

$$K^1 = \{(\text{Max}\{\Pi_1(q_1, Q_2(q_1)), 0\}, \text{Max}\{\Pi_2(q_1, Q_2(q_1)), 0\}) : D^{-1}(q_1+Q_2(q_1)) > 0\},$$

where the firms do not participate if their profits are negative. We can similarly define Q_1 and K^2.

We give three examples to show the kinds of results which can be obtained by using the negotiation model. These results are shown in Figures 5 to 6, which are in each case the graphs of the sets K^1 and K^2, obtained by computer calculation. In these figures the axes are omitted, since they contain some relevant parts of the sets, and the sets has been slightly displaced for purposes of distinction. The functions f and g and the negotiation equilibria in each figure can be obtained by inspection from the graphs of K^1 and K^2 respectively.

The indirect demand function used in each example is $p = 2.0 - 0.5\,q$.

Example 1. The firms have production costs

$$C_1(q_1) = 0.1 + 0.045\,q_1,\quad C_2(q_2) = 0.4 + 0.01\,q_2.$$

There are two perfect equilibrium given in Figure 5 by points A and B. To see that B is an equilibrium observe that for small τ, $0 \in S^2_\tau$ (writing 2 instead of M for the second player) because $0 \in P^2$, $f(0) \geq 0$, and $\mu g(\rho f(0)) = 0$. This holds independently of whether f(0) is identical to u^+. This means that even if profits are slightly different for Firm 1 if it announces production at the optimal monopoly level *before* Firm 2 makes any announcement, then if it does so *after* Firm 2 announces zero participation, B is still an equilibrium point.

Example 2. The firms have identical production costs

$$C^1_i(q_i) = \mathrm{Min}[0.1 + 0.50\,q_i;\;\; 0.5 + 0.05q_i]$$

which represent a production function in which for larger scales a technology offering substantial savings is available. The equilibria are given in Figure 6 by points C and E (similar to point B in Example 1), which represent monopoly production by one of the firms, using the larger scale technology. Point D, a Cournot equilibrium in the sense that each Firm is maximizing profits given the other firm's level of production, is not an equilibrium because it is not a lower boundary point of the sets $\Sigma^M{}_0$, $\Sigma^J{}_0$.

Example 3. The firms have identical production costs

$$C^1_i(q_i) = \mathrm{Min}[0.50\,q_i;\;\; 0.5 + 0.05q_i]$$

which represent the same production function as in Example 2, except that the lower scale technology has zero fixed costs. The equilibria are given in Figure 7 by points F and H. In each case one of the firms uses the large scale technology and obtains larger profits, while the other uses the low scale technology and has lower profits. Point G, again Cournot, is not an equilibrium for the same reason.

The general two-agent negotiation game

We solve for the subgame perfect Nash equilibria of the game G^∞ introduced in section II.

Consider such a perfect equilibrium of G^∞. Since the game is identical at each occurrence of G^J or G^M, so is the maximum current value of the game. Therefore, to avoid the discount in utility, both players will end the game at the first opportunity. This means that the game ends at the first occurrence of G^J or G^M. Therefore we need only explore strategic situations involving sequences of payoffs which are stationary for the proponents. Thus, let (u_1, v_1), ... (u_4, v_4), be equilibrium proposals in the first to fourth periods. Then $u_3 = u_1$ and $v_2 = v_4$. However, we cannot assume $v_3 = v_1$ or $u_4 = u_2$.

Let the set of responses to proposals be $\{\mathbf{a}, \mathbf{r}, \mathbf{u}\}$, meaning "accept", "reject" and "opt for reserve utility". In equilibrium the players will accept proposals if they are worth more than the present value of the game if they reject or take their reserve utility. Mary's strategy for responding to proposals in O^J, is a function $\phi^M : O^J \to \{\mathbf{a}, \mathbf{r}, \mathbf{u}\}$. If the present value of the game when Mary decides to reject to make a proposal is υ, Mary's response strategy is

$$\phi^M(u_1, v_1) = \mathbf{a} \text{ if } v_1 \geq \max\{\upsilon, v_R\},$$

$$\phi^M(u_1, v_1) = \mathbf{r} \text{ if } \upsilon > \max\{v_1, v_R\},$$

$$\phi^M(u_1, v_1) = \mathbf{u} \text{ if } \max\{v_1, \upsilon\} < v_R.$$

If $\upsilon \geq v_R$ the strategy **u** is not used and the relation between υ and v_1 determines the response, while if $\upsilon < v_R$ it only depends on the relation between v_1 and v_R. Thus we can first study the value of the game when the players do not have the reserve utility as an option. Mary uses the strategy

$$\phi^M(u_1, v_1) = \mathbf{a} \text{ if } v_1 \geq \upsilon,$$

$$\phi^M(u_1, v_1) = \mathbf{r} \text{ if } \upsilon > v_1;$$

Joan uses the analogous strategy

$$\phi^J(u_2, v_2) = \mathbf{a} \text{ if } u_2 \geq \omega,$$

$$\phi^J(u_2, v_2) = \mathbf{r} \text{ if } \omega > u_2,$$

where in equilibrium if Mary accepts a proposal (u_1, v_1), $\upsilon = v_1$ and $\omega = \rho u_1$, because Joan would abandon her proposal u_1 if she knew she would receive a proposal worth ω/ρ in the next round, while if Joan accepts a proposal (u_2, v_2) $\omega = u_2$ and $\upsilon = \mu v_2$.

Given these strategies, what is Joan's optimal strategy in the case in which she makes a proposal which is to be accepted? Although Joan and Mary have announced strategies ω and υ, they still examine the possibility of deviating to obtain a better payoff. Consistency will give us the equilibrium conditions for ω and υ below. We characterize the set of proposals (u_1, v_1) which Joan can make yielding her u_1 and which Mary will accept. These proposals will end the game in the first period. Mary accepts an offer $v_1 \geq \upsilon$ if in period 2 when she makes a proposal she cannot obtain $v_2 > (1/\mu)v_1$ and still offer Joan $u_2 \geq \rho u_1 \geq \omega$ (both inequalities are derived from the principle that Joan will accept an offer only if it is worth more than the value of the game if she rejects). In other words, Mary will accept a proposal if (a) $v_1 \geq \upsilon$; (b) $\rho u_1 \geq \omega$; and (c) if $u_2 \geq \rho u_1$ is a feasible proposal for Mary, then $g(u_2) \leq (1/\mu)v_1$. (c) is equivalent to $g(\rho u_1) \leq (1/\mu)v_1$ because g is decreasing. We shall refer to conditions (a), (b) and (c) throughout the paper. The set of these proposals is

$$O^J = \{(u_1, v_1) \in K^J \mid v_1 \geq \upsilon, \rho u_1 \geq \omega, \text{ and } \mu g(\rho u_1) \leq v_1\}.$$

This is the set of proposals available to Joan which give her at least ω/ρ and which Mary cannot Pareto improve.

Define O^M symmetrically to O^J. We analyze now the process of maximization within the proposal sets O^J and O^M. First observe that $(u_1, v_1) \in O^J \Rightarrow \exists\, v_1' \geq v_1 :$ $(f(v_1'), v_1') = (f(v_1), v_1') \in O^J$ (because of the inequalities $f(v_1') = f(v_1) \geq u_1 \geq$

ω/ρ, $\mu g(\rho f(v_1)) \leq \mu g(\rho u_1) \leq v_1 \leq v_1')$, and that $f(v_1)$ is the maximum value that Joan can obtain if she offers v_1 to Mary. Therefore finding the maximum value for Joan among the proposals O^J,

$$U^*_1 = \max\{u_1 \mid \exists\ v_1 : (u_1, v_1) \in O^J\},$$

is equivalent to finding the maximum value in the subset of proposals

$$o^J = \{(f(v), v) \in K^J \mid v_1 \geq \upsilon,\ \rho u_1 \geq \omega, \text{ and } \rho f(v) \leq u^+ \Rightarrow \mu g(\rho f(v)) \leq v\},$$

Let

$$S^M = \{v \in [\upsilon, v^+] \mid \rho f(v) \geq \omega \text{ and } \rho f(v) \leq u^+ \Rightarrow \mu g(\rho f(v)) \leq v\}.$$

We continue referring to the three conditions for v to belong to S^M as conditions (a), (b) and (c). Note that although this set still refers to Joan's strategic proposals, we have indexed it with M because it is a set of payoffs for Mary. From the definition of S^M it follows that

$$o^J = \{(f(v), v) \mid v \in S^M\}$$

and

$$U^*_1 = f(\inf(S^M)).$$

The following Proposition proves that if S^M is non-empty, then it is closed below.

Proposition 1 Suppose that $S^M \neq \varnothing$ and let v_0 be any lower boundary point of S^M (that is, $\forall\ \varepsilon > 0\ \exists\ v \in S^M : v_0 \leq v < v_0 + \varepsilon$ and $\exists\ \varepsilon > 0 : (v_0 - \varepsilon) \cap S^M = \varnothing$).

(1) $v_0 \in S^M$.

(2) If $v_0 > \upsilon$ then $\mu g(\rho f(v_0)) = v_0$.♦

Let

$$v^*_1 = \inf(S^M).$$

Proposition 1 implies

$$U^*_1 = f(v^*_1).$$

To obtain this value Joan can make a proposal (U^*_1, V^*_1) in which she can choose any

$$V^*_1 \in \Phi^M = \{v \in [v^*_1, v^+] \mid (U^*_1, v) \in K^J\}.$$

Φ^M consists of those values that Joan can offer Mary while still obtaining U^*_1. Φ^M has the following properties, which guarantee that these values exist and are in S^M.

Proposition 2 Suppose $S^M \neq \varnothing$.

(1) $\Phi^M \neq \varnothing$;

(2) $\Phi^M \subseteq [v^*_1, v^+_1] \subseteq S^M$, where $v^+_1 = \sup\{v \in [v^*_1, v^+] \mid f(v) = f(v^*_1)\}$, and $v^+_1 \in \Phi^M$.♦

We analogously define S^J to analyze Mary's proposals:

$$S^J = \{u \in P^J \mid \mu g(u) \geq \upsilon \text{ and } \mu g(u) \leq v^+ \Rightarrow \rho f(\mu g(u)) \leq u\}.$$

where

$$P^J = [\omega, u^+].$$

As in the case of Joan, we define

$$u^{\#}_2 = \inf(S^J), \quad V^{\#}_2 = g(u^{\#}_2),$$

$$\Phi^J = \{u \in [u^{\#}_2, u^+] \mid (u, V^{\#}_2) \in K^J\}.$$

where the symbol # denotes current values. The following Propositions are equivalent to Propositions 1 and 2.

Proposition 3. Suppose that $S^J \neq \varnothing$ and let u_0 be any lower boundary point of S^J.

(1) $u_0 \in S^J$.

(2) If $u_0 > \omega$ then $\rho f(\mu g(u_0)) = u_0$. ♦

Proposition 4. Suppose that $S^J \neq \varnothing$.

(1) $\Phi^J \neq \varnothing$;

(2) $\Phi^J \subseteq [u^{\#}_2, u^+_2] \subseteq S^J$, where $u^+_2 = \sup\{u \in [[u^{\#}_2, u^+] \mid g(u) = g(u^*_2)\}$, and $u^+_2 \in \Phi^J$.

If Mary can make an offer in the second period which will be accepted by Joan, it will be $(U^{\#}_2, V^{\#}_2)$, for some $U^{\#}_2 \in \Phi^J$. The present value of this offer in period 1 is: $(U^*_2, V^*_2) = (\rho U^{\#}_2, \mu V^{\#}_2)$. Summarizing, the optimal proposal for a player if the other is to have a given response is in Table 1:

Other player's response	a	r
Joan's optimal proposal	$(u_1, v_1) = (U^*_1, V^*_1)$	$(u_1, v_1) \notin O^J$
Mary's optimal proposal	$(u_2, v_2) = (U^{\#}_2, V^{\#}_2)$	$(u_2, v_2) \notin O^M$

Table 1. Optimal proposal for each player to obtain the other player's given response.

Since some proposals are made for the other player to reject them and make her own, we define the concept of declining.

Definition 4 Joan (respectively Mary) is *able to decline* if the set $K^J \setminus O^J$ (respectively $K^M \setminus O^M$) is non-empty. She *declines* if she makes a proposal in this set. ♦

Since if Joan makes a proposal which Mary accepts it will be one of the proposals (U^*_1, V^*_1), $V^*_1 \in \Phi^M$, while if she declines the result does not depend on which proposal in $K^J \setminus O^J$ she uses, we shall abbreviate making a proposal with **P**, declining with **D**. We use the same notation for Mary.

Table 1 implies that a player's response to **P** is **a** (accepting the offer) and to **D** is **r** (rejecting the offer to make a proposal). If a player responds with **r** then her proposal will be **P** or **D**. Therefore, when it is their turn to make a proposal in a perfect strategy, the players choose among the strategic options **P**, **D**, and when they receive a proposal **D**, they choose between the responses **P**, **D** (meaning **r** followed by **P** or **D**). Therefore if the game begins with Joan, the extended form of

G^∞ reduces to G', while if it begins with Mary it reduces to G'' (see Figure 2). In each case it is understood that the option **P** only exists if the corresponding set S^M or S^J is non-empty, and the option **D** only exists if the corresponding player can decline. Since the subgame perfect equilibria of G^∞ are those which are Nash independently of the node at which the game starts, we consider the Nash equilibria of the games G' and G'', which correspond to starting at nodes 1 and 2 (see Figure 1). Considering games played beginning at nodes of types 1′ or 2′ does not add further restrictions since each player responds to the given proposal according to whether it is of type **P** or **D** independently of whether it is optimal for the proponent.

Observe that Joan may have the strategy of offering (U^*_1, V^*_1) in the first period and $(U^*_1, V^*_1{}')$ in the second if the set V has more than one element, while Mary may similarly offer $(U^\#_2, V^\#_2)$ in the first period and $(U^\#_2{}', V^\#_2)$ in the second if the set U has more than one element. These strategies do not obtain better payoffs for the proponent but may make a difference to the comparisons of the other player. However, since the game ends in the first two periods only Joan's second proposal can make a difference.

The normal forms of G' and G'' are given in Tables 2 and 3.

	P $(S^J \neq \varnothing)$	**D** $(O^M \neq K^M)$
P $(S^M \neq \varnothing)$	(U^*_1, V^*_1)	(U^*_1, V^*_1)
D $(O^J \neq K^J)$	(U^*_2, V^*_2)	$(0, 0)$

Table 2. Normal form of the game G'.

	P $(S^J \neq \varnothing)$	**D** $(O^M \neq K^M)$
P $(S^M \neq \varnothing)$	(U^*_2, V^*_2)	$(\rho^2 U^*_1, \mu^2 V^*_1{}')$
D $(O^J \neq K^J)$	(U^*_2, V^*_2)	$(0, 0)$

Table 3. Normal form of the game G''.

The following lemma describes how the sets S^J, S^M are interrelated from the strategic point of view. First, only one of the players may have proposals which the other cannot Pareto improve (Pareto unimprovable proposals); this player will dominate the negotiation game. Second, if both players have such proposals, then there are two cases. In case A each player prefers her own Pareto unimprovable proposals, and in case B each player weakly prefers the other player's Pareto unimprovable proposals. The inequalities will be used to find the Nash equilibria of games G' and G''. The definition provides nomenclature for the lemma.

Definition 5 In what follows we shall say that the values u_1, u_2, v_1, v_2 form a *quadruple* if $v_1 \in S^M$, $u_1 = \rho f(v_1)$, $u_2 \in S^J$, $v_2 = \mu g(u_2)$.♦

Lemma 1

(1) Suppose that only one of the sets S^J, S^M is non-empty.

(1.1) If $S^J = \varnothing$ and $S^M \neq \varnothing$ then $\rho f(v^*_1) \geq \omega \Rightarrow v^*_1 = \upsilon$.

(1.2) If $S^M = \varnothing$ and $S^J \neq \varnothing$ then $\mu g(u^{\#}_2) \geq \upsilon \Rightarrow u^{\#}_2 = \omega$.
(2) Suppose that both sets S^J, S^M are non-empty.
(2.1) Let u_1, u_2, v_1, v_2 be a quadruple of values. Then

$$u_1 \leq u_2 \iff v_1 \geq v_2.$$

(2.2) The conditions

$$\begin{aligned} &\exists\, u_2 \in S^J, u_1 \in \rho f(S^M) : u_1 > u_2, \\ &\exists\, v_1 \in S^M, v_2 \in \mu g(S^J) : v_1 < v_2, \end{aligned} \tag{A}$$

imply each other and imply

$$\rho U^*_1 > u^{\#}_2 \text{ and } \mu V^{\#}_2 > v^*_1.$$

(2.3) In the case opposite to *(2.2)*, that is, if one of the hypothesis

$$\begin{aligned} &\rho f(S^M) \leq S^J, \\ &\mu g(S^J) \leq S^M, \end{aligned} \tag{B}$$

holds (they imply each other) then

$$\rho U^*_1 \leq u^{\#}_2 \text{ and } \mu V^{\#}_2 \leq v^*_1.$$

(2.4) If hypothesis (A) hold, then

$$\rho U^*_1 > U^{\#}_2 \geq u^{\#}_2 \text{ and } \mu V^{\#}_2 > V^*_1 \geq v^*_1,$$

(2.5) If hypothesis (B) hold, then

$$\rho U^*_1 \leq u^{\#}_2 \leq U^{\#}_2 \text{ and } \mu V^{\#}_2 \leq v^*_1 \leq V^*_1.$$

(2.6) Summarizing,

$$\rho^2 U^*_1 > U^*_2 \Leftrightarrow V^*_1 < V^*_2 \Leftrightarrow \text{hypothesis (A)},$$

$$\rho^2 U^*_1 \leq U^*_2 \Leftrightarrow V^*_1 \geq V^*_2 \Leftrightarrow \text{hypothesis (B)}.$$

(3) In the cases when ω and υ are consistent we have:
(3.1) Assume $\rho U^*_1 = \omega$, $V^*_1 = \upsilon$. Then $S^M = [\upsilon, v^+_1]$. Either $\mu g(\omega) < \upsilon$ and $S^J = \varnothing$ or $\mu g(\omega) = \upsilon$ and $S^J = [\omega, u^+_2]$.
(3.2) Assume $V^*_2 = \mu V^{\#}_2 = \upsilon$, $U^*_2 = \mu U^{\#}_2 = \mu\omega$. Then $S^J = [\omega, u^+_2]$. Either $\rho f(\upsilon) < \omega$ and $S^M = \varnothing$ or $\rho f(\upsilon) = \omega$ and $S^M = [\upsilon, v^+_1]$.
(3.3) If either one of the consistency assumptions hold, and both of the sets S^M, S^J are non-empty, then $\rho f(\upsilon) = \omega$, $\mu g(\omega) = \upsilon$, and $V^*_1 = V^*_2$ and $U^*_1 > U^*_2$.♦

We define now some additional strategic set.

Definition 6 Let

$$\Theta^J = \{(u_1, v_1) \in K^J \mid v_1 \geq v_R, u_1 \geq u_R, \text{ and } \mu g(\rho u_1) \leq v_1\},$$

$$\Sigma^M = \{v \in [v_R, v+] \cap K^J(v) \mid f(v) \geq u_R \text{ and } \rho f(v) \leq u^+ \Rightarrow \mu g(\rho f(v)) \leq v\},$$

$$\Xi^M = \{v \in \Sigma^M \mid \mu g(\rho f(v)) = v\},$$

$$\Gamma^M = \{v \in \Sigma^M \mid [\mu^2 v, v) \cap \Sigma^M = \varnothing\}.$$

Θ^M, Σ^J, Ξ^J, Γ^J are defined symmetrically.♦

The set Γ^M is a subset of the lower boundary points of Σ^M. Since $\Sigma^M \subseteq S^M$, some of these may be lower boundary points of S^M, which have been characterized in Proposition 1(2). We characterize the remaining points.

Proposition 5 Let $v \in \Gamma^M$. If $v > \upsilon$ then either $\mu g(\rho f(v)) = v$ or v is on a segment containing $[\mu^2 v, v)$ on which f is constant.♦

As will be understood from the Lemma below, Θ^J is the sets of Joan's proposals which in principle may be accepted. Σ^M represents the region of the efficiency boundary on which Joan is weakly dominant. Ξ^M represents those proposals of Σ^M at which Joan locally maximizes her utility and which Mary can replicate. Γ^M represents those points of Σ^M at which there are no nearby proposals which can give Mary an incentive to deviate from her response strategy υ. We examine the perfect equilibria of G^∞.

Lemma 2 Suppose Joan (respectively Mary) has the strategy to accept if offered at least ω (respectively υ), otherwise to reject and propose (U^*_1, V^*_1) (respectively $(U^\#_2, V^\#_2)$). If the payoffs of G' are (U^*_1, V^*_1) then consistency means $\rho U^*_1 = \omega$ and $V^*_1 = \upsilon$. These are better than the reserve utility if $\omega \geq u_R/\rho$, $\upsilon \geq v_R$. If the payoffs are are (U^*_2, V^*_2), then consistency means $U^\#_2 = \omega$ and $\mu V^\#_2 = \upsilon$. These are better than the reserve utility if $\omega \geq u_R/\mu$, $\upsilon \geq v_R$. We will assume in each case that the strategies are consistent with the payoffs and that these are greater than the reserve utilities.

(1) If both sets S^J, S^M are empty, the value of the game is (0, 0).

(2) If just S^M is non-empty, **(P, D)** is the only strategic combination which is Nash in the games G' and G''. It has payoffs (U^*_1, V^*_1) in G'. The players do not have incentives to deviate to lower response values when beginning the game at any of the nodes 1′ and 2′, that is, when responding to any proposal, if and only if $V^*_1 \in \Gamma^M$.

(3) If just S^J is non-empty, **(D, P)** is the only strategic combination which is Nash in the games G' and G''. It has payoffs (U^*_2, V^*_2) in G'. The players do not have incentives to deviate to lower response values if and only if $U^\#_2 \in \Gamma^J$.

(4) If both sets S^J, S^M are non-empty, an equilibrium in which neither player has an incentive to deviate from the response strategies only exists under hypothesis B with the properties $\rho f(\upsilon) = \omega$, $\mu g(\omega) = \upsilon$, $V^*_1 = V^*_2$, $U^*_1 > U^*_2 \geq \rho^2 U^*_1$, $V^*_1 \in \Xi^M$, $V^*_2 \in \Xi^J$. There are two cases.

(4.1) $V^*_2 \geq \mu^2 V^*_1{}'$. **(P, P)** is the only strategic combination which is Nash in the games G' and G''. It has payoffs (U^*_1, V^*_1) in G'. The players have no incentives to deviate from their response strategies.

(4.2) $V^*_2 < \mu^2 V^*_1{}'$. Consistency means $\rho U^*_1 = \omega$ and $V^*_1 = \upsilon$. The players have no incentives to deviate from their response strategies if and only if $V^*_1 \in \Gamma^M$.♦

Summarizing:

Theorem 3

(1) Let $\Gamma^{J\prime} = \{u \in \Gamma^J \mid \mu g(u/\rho) \notin \Xi^M\}$.

There is a one to one correspondence between the set $\Gamma^M \cup \Xi^M$ and the subgame perfect equilibria payoffs of G^∞ in which the accepted proposal is Joan's and between the set $\Gamma^{J\prime}$ and the equilibria payoffs in which the accepted proposal is Mary's, except if both sets are empty in which case the games is worth (u_R, v_R).

(2) If Σ^J is empty and Γ^M consists of a single point (this is a consequence if f is decreasing), then G^∞ is equivalent to a principal agent game in with Joan as principal (see Table 4).

	a
P	(U^*_1, V^*_1)

Table 4. When $\Sigma^J = \varnothing$ and Γ^M is a singleton the game reduces to a principal agent game.

Here $U^*_1 = f(v_R)$.

(3) If Σ^M is empty and Γ^J consists of a single point (this is a consequence if g is decreasing), the game takes the form

	P	u
D	(U^*_2, V^*_2)	(u_R, v_R)
U	(u_R, v_R)	(u_R, v_R)

Table 5. When $\Sigma^M = \varnothing$ and Γ^J is a singleton the game reduces to a principal agent game only if $U^*_2 \geq u_R$ and $V^*_2 \geq v_R$.

Here $V^*_2 = g(u_R)$. In this game Joan can take her reserve utility or decline (by making any proposal) and play a game in which Mary is a principal agent. Joan will decline only in the unusual case $U^*_2 \geq u_R$, and then Mary will reject to offer (U^*_2, V^*_2) only if $V^*_2 \geq v_R$. This situation is non-trivial only in special cases in which Mary's best strategy for herself does not imply offering the reserve utility to Joan.♦

The negotiation game can become a Principal Agent game

As Theorem 3 shows, when the first player has a Pareto improvement over any of the other player's proposals, the negotiation game becomes a principal agent game in her favor. We complete the study of this case by letting the players decide who makes the first offer. Let $G^{\infty\prime}$ be the game G^∞ with the roles of the players interchanged, so that Mary makes the first proposal and let H be the game with normal form

	Mary wants Joan to begin	Mary wants to begin
Joan wants to begin	play G^{∞}	(u_R, v_R)
Joan wants Mary to begin	(u_R, v_R)	play $G^{\infty\prime}$

Table 6. Normal form of the game H.

The game H, which is played instantly (without causing a discount in utility), consists of deciding who makes the first proposal in the negotiation game. If both players agree (they issue their decisions simultaneously) then the corresponding game is played. If they disagree, then they each obtain their reserve utility. We have the following theorem.

Theorem 4 If Σ^M is empty and Γ^J consists of a single point or if Σ^J is empty and Γ^M consists of a single point then Joan and Mary agree on who should make the first proposal. In the first case Joan begins and H is equivalent to the principal agent game shown in Table 5, while in the second Mary begins and H is equivalent to the game in Table 7 (in which case if Mary's proposals are sufficiently productive, so $\mu g(u^{\#}_2) \geq v_R$, $V^{\#}_2 = g(u_R)$).♦

	P
a	$(U^{\#}_2, V^{\#}_2)$

Table 7. Outcome of H when only S^J is non-empty.

The outcome when ρ and μ tend to zero

In the general the strategies of the game H reflect the coordination problems present in games G^{∞} and $G^{\infty\prime}$, because of the possibilities of multiple equilibria. Instead of examining these cases we prefer to follow Rubinstein and examine what happens when we set $\rho = e^{-\alpha\tau}$, $\mu = e^{-\beta\tau}$ and let τ tend to zero. Write a suffix τ on each of the strategic sets we have defined for the case $\tau > 0$: S^M_τ, Θ^J_τ, etcetera. The limit equilibria are given by the limit sets of $\Gamma^M_\tau \cup \Xi^M_\tau$ and $\Gamma^{J\prime}_\tau$. Recall Definitions 1, 2 and 3. We calculate these in the non-pathological cases in which the sets Σ^M_0, E^M_0, F^M_0, Γ^M_0 defined in section II and Γ^M_τ have a common bound N to the number of their boundary points. Further, we shall need the limits of the sets Φ^M_τ, Φ^J_τ defined above (see Propositions 2 and 4).

Definition 7 Let Φ^M_0 (respectively Φ^J_0) be the set of points each of which is the limit of some sequence of points $u_\tau \in \Phi^M_\tau$ (respectively $v_\tau \in \Phi^J_\tau$).♦

The following lemma establishes the conditions for Joan to have a proposal which Mary will accept in a limit equilibrium in terms of the properties of f and the sets $\Sigma^M{}_0$, $E^M{}_0$, $F^M{}_0$, $\Gamma^M{}_0$. The analogous statements hold for Mary's proposals. The lemma is applied in the proof of Theorem 1. We shall say $f^\varepsilon(v)$ satisfies a property arbitrarily near on the left (right) of v_0 if $v < v_0$ can be found for which the property is true arbitrarily near v_0.

Lemma 3

(1) If $v \in \Sigma^M{}_0$, and $f(v) > u_R$ if $u_R > 0$, then for sufficiently small τ, $v \in \Sigma^M{}_\tau$.

(2) If $v \in F^M{}_0$, then for sufficiently small τ, $v \notin \Sigma^M{}_\tau$.

(3) The maps $f : E^M{}_0 \to E^J{}_0$, $g : E^J{}_0 \to E^M{}_0$ are inverses where the composition is defined. If v_0 is an interior point of $E^M{}_0$ f is continuous at v_0.

(4) Let $u_0 = f(v_0)$ and suppose that u_0 and v_0 are interior points of $E^J{}_0$, $E^M{}_0$ respectively, and that f has a continuous first derivative at v_0. If there are values u_τ, v_τ tending to u_0, v_0 as τ tends to zero satisfying $u_\tau = \rho_\tau f(v_\tau)$, $v_\tau = \mu_\tau g(u_\tau)$ then

$$f^\varepsilon(v_0) = \alpha/\beta \text{ and equivalently } g^\varepsilon(u_0) = \beta/\alpha.$$

For the remaining points suppose that f and g are in the PC^1 class of functions and that the sets $\Sigma^M{}_0$, $E^M{}_0$, $F^M{}_0$, $\Gamma^M{}_0$ defined in section II and $\Gamma^M{}_\tau$ have a common bound N to the number of their boundary points.

(5) Suppose for all small enough τ, $v_\tau \in \Gamma^M{}_\tau \cup \Xi^M{}_\tau$. Then the limit $v_0 = \lim_{\tau\to 0} v_\tau$ exists and is in the set $\Gamma^M{}_0 \cup E^M{}_0$.

(6) Let $h(\tau, v) = \mu g(\rho f(v))/v$. Then

$$h(\tau, v) = h(0, v)\exp[\int_0^\tau \{-\beta + \alpha g^\varepsilon(\rho f(v))\}d\tau].$$

(7) Suppose v_0 satisfies LHC1. Then $\Sigma^M{}_\tau \cap (v_0-\varepsilon, f^{-1}(\rho^{-1}f(v_0))) = \varnothing$ for small enough τ.

(8) Suppose v_0 satisfies RHC1. Then $\Sigma^M{}_\tau \cap [v_0, v_0+\varepsilon) \neq \varnothing$ for small enough τ.

(9) Let $v_0 \in E^M{}_0$. There exists an $\varepsilon > 0$ such that for sufficiently small values of τ there are points $v_\tau \to v_0$ which are the infimae of $\Sigma^M{}_\tau \cap I$, where $I = (v_0-\varepsilon, v_0+\varepsilon)$, if and only if v_0 satisfies one of the left hand and one of the right hand conditions.

(10) If $v_0 \in \Gamma^M{}_0 \setminus E^M{}_0$ then v_0 is the limit of a sequence $v_\tau \in \Gamma^M{}_\tau$.

(11) $\Phi^M{}_0 \neq \varnothing$;

(12) $\Phi^M{}_0 \subseteq [v^*, v^+{}_1]\}.\subseteq E^M{}_0.\cup \Sigma^M{}_0$, where $v^+{}_1 = \sup\{v \in [v^*, v^+] \mid f(v) = f(v^*{}_1)\}$; $v^+{}_1 \in \Phi^M{}_0$.

(13) The direct definition $\Phi^M{}_0 = \{v \in [v^*, v^+] \mid (U^*{}_1, v_R) \in K^J\}$, which does not pass through the limit process, holds if $v^* \in \Gamma^M{}_0$ or if v^* is an interior point of $E^M{}_0$. ♦

We have not calculated the set $\Phi^M{}_0$ in the case where v^* is a boundary point of $E^M{}_0$.at which f has a constant portion. This involves examining the path of convergence of v^* and is not complicated in particular examples. However, the calculation in general involves enough cases not to be warranted here.

The reserve utility and the Nash Product

In the case where the negotiation equilibrium occurs in the interior of E^{M}_0 at a point at which $f^{\varepsilon}(v_0) = \alpha/\beta$, Binmore (1994 section 5.5.6) shows that the equilibrium can be seen as occurring at the tangential intersection of the efficiency set with a level set of the Nash product $NP = (u - u_0)^{\alpha}(v - v_0)^{\beta}$. u_0 and v_0 are not defined there but are sometimes referred to as "points of conflict", and others as "reserve utilities".

Note in the first place that negotiation equilibria in general need not occur at some point at which the Nash product is maximized. In our results, equilibria are first explained by Pareto dominance and then by the relative impatience of the players as expressed by the quotient α/β, which only distinguishes proposals in cases in which neither of the players has Pareto dominance. Thus perhaps the best interpretation of the quotient α/β is a purely economic one relating to each players' surrounding context (as the rate of return of alternative projects) or time preferences, rather than as an index of power, whose effects should be understood as intervening in the proposal sets.

If the function NP represents some kind of social welfare function which is the result of the relative power of the players, u_0 and v_0 represent levels of utility each player is prepared to the death not to accept. They are never reached as the result of negotiation. Thus they cannot represent the behavior associated with reserve utility. If negotiation cannot overcome the tiny loss of utility near these values, implicitly a large amount of power must be used to prevent the realization of these "points of conflict". But such power must cost, and therefore the original utility units are not representing the psychological or economic costs involved. This again supports the notion that power and preferences must be reflected in the proposal sets.

Final Remarks

We have shown that Rubinstein's negotiation paradigm can be extended not only to consider non-convex Pareto efficiency sets but efficiency sets originating in arbitrary (compact) proposal sets in which different regions belong to different players. The concept that underlies the logic of this Paradigm is the concept of regions of Pareto dominance. In our results, equilibria are first explained by Pareto dominance and then by the relative impatience of the players as expressed by the quotient α/β, which only distinguishes proposals in cases in which neither of the players has Pareto dominance.

In some additional equilibria, discontinuities in the efficiency sets allow the weaker player to maintain a perfect equilibrium with higher utility than she would achieve if the Pareto dominant player maximized her utility in the connected component of the Pareto region in which the equilibrium proposal is found. However these equilibria do not have the property that the accepting player can propose them herself. This concept may be added to the subgame perfect concept to eliminate these equilibria in contexts in which they are thought to be unrealistic.

In some case the multiple equilibria would disappear in a wider game with more players in which one of the negotiating player's demands for a higher payoff could be untenable. This would be the case if there were many negotiating pairs in a context of demand and supply. In others the multiplicity of equilibria leaves the door open for further characterizations of the agent's behavior, be they economic or psychological. An extension to the general case of our example of negotiation when the players have moral beliefs in equitable distribution would tend to reduce the number of equilibria.

We have also shown some additional facts. If the efficiency set corresponds to one player's proposals, then the negotiation game becomes a Principal-Agent game. Conversely, if several agents, who are negotiating individually with a principal, organize, they may change their situation back to one of bilateral negotiation. In the case of such examples as duopoly, some of the Nash (or Cournot) equilibria become unstable in the context of a negotiation game (which may represent a sequence of actions or games rather than negotiation per se) and are not perfect equilibria.

Finally, another implication of the existence of multiple equilibria is that negotiated solutions to problems of agreement are not always possible. It may happen that each player prefers her own or the other player's proposal. One or the other player will be the looser at any equilibrium, and one must interpret that negotiation breaks down. Some of these problems may be solved if a third party inserts them in a wider context or game, or, again, if some moral or legal principles of distribution (supported by the context) prevail.

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Appendix

The definitions, propositions, lemmas and theorems are numbered in the order of their presentation in the text. They are proved in their logical order.

Proof of Proposition 1.

(1) Let v_k be a decreasing monotonic sequence in S^M tending to v_0. (a) $v_0 \in [\upsilon, v^+]$; (b) since $\rho f(v_k) \geq \omega$ and f is decreasing, $\rho f(v_0) \geq \omega$; and (c) if $\rho f(v_0) \leq u^+$, also $\rho f(v_k) \leq u^+$, so that $\mu g(\rho f(v_k)) \leq v_k$. Now, $\rho f(v_k)$ is increasing and bounded, so it tends to a limit $\lim_k \rho f(v_k) \leq \rho f(v_0)$ and as g is left continuous, $\mu g(\rho f(v_0)) \leq \mu g(\lim_k \rho f(v_k)) \leq v_0$.

(2) Suppose $v_0 > \upsilon$. We first show that $\rho f(v_0) \le u^+$. For if this is not true then for small enough δ, $v_0 - \delta$ satisfies conditions (a) and (b), and also (c), since $\rho f(v_0 - \delta) \ge \rho f(v_0) > u^+$. Thus $v_0 - \delta \in S^M$, contradicting the definition of v_0. Next, as $v_0 \in S^M$ implies $\mu g(\rho f(v_0)) \le v_0$, we show that the strict inequality cannot hold. Suppose for contradiction that $\mu g(\rho f(v_0)) < v_0$. For small enough δ, $\mu g(\rho f(v_0)) < v_0 - \delta$, and $v_0 - \delta$ satisfies conditions (a) and (b). Now, if $\rho f(v_0 - \delta) \le u^+$, $\mu g(\rho f(v_0 - \delta)) \le \mu g(\rho f(v_0)) < v_0 - \delta$, so (c) holds and $v_0 - \delta$ is in S^M.♦

Proof of Proposition 2.
(1) $f(v^*_1) = U^*_1 \Rightarrow \exists\, v \in [v^*_1, v^+] : (U^*_1, v) \in K^J$ and $f(v) = f(v^*_1) = U^*_1$, so that $v \in \Phi^M$.
(2) Since f is left continuous $f(v^+) = f(v^*_1)$. It is easy to verify $[v^*_1, v^+_1] \subseteq S^M$. If $v \in \Phi^M$, $f(v) \ge U^*_1 = f(v^*_1)$ because $(U^*_1, v) \in K^J$, while $f(v) \le f(v^*_1)$ because f is decreasing, so $v \in [v^*_1, v^+_1]$.♦

The following lemma is of a technical nature, establishing relations of order among the values U^*_1, U^*_2, u_R and V^*_1, V^*_2, v_R, relations between properties of f and g, and the emptiness or non-emptiness of S^J and S^M. It will be used in the proofs below.

Lemma 4
(1) $f(v_R) < u_R \Rightarrow S^M = \varnothing$ and $g(u_R) < v_R \Rightarrow S^J = \varnothing$.
(2) $g(u_R) \ge v_R \Rightarrow u_R \le u^+$; $f(v_R) \ge u_R \Rightarrow v_R \le v^+$.
(3) If both of the sets S^J, S^M are empty then $\rho f(v_R) < u_R$ and $\mu g(u_R) < v_R$.
(4) There exists the case in which S^J, S^M are empty and
$\rho f(v_R) < u_R < f(v_R)$ and $\mu g(u_R) < v_R$. (∗)
(5) There exists the case in which S^J, S^M are non-empty, inequalities (∗) hold, and $u^\#_2 > u_R$, $v^*_1 > v_R$.
(6) Suppose $U = \{u^\#_2\}$ and f is continuous at V^*_2. Then if $S^M = \varnothing$ and $S^J \ne \varnothing$, $U^*_2 \ge u_R$ and $V^*_2 \ge v_R$ cannot both hold.
Proof
(1) This is clear.
(2) $g(u_R) \ge v_R \Rightarrow \exists\, (u,v) \in K^J$ such that $u \ge u_R$, $v \ge v_R$, so $u_R \le u^+$. The other implication is obtained similarly.
(3)(i) We show that $\mu g(u_R) \ge v_R$ and $S^J = \varnothing \Rightarrow S^M \ne \varnothing$. Using *(2)*, $\mu g(u_R) \ge v_R$ and $S^J = \varnothing \Rightarrow \exists\, u \in [u_R, u^+] : \mu g(u) \le v^+$ and $\rho f(\mu g(u)) > u$. We show that $v = \mu g(u)$ is in S^M. We already have:
(a) $v_R \le v \le v^+$; (b) $f(v) > (1/\rho)u \ge u_R$; and (c) if $\rho f(v) \le u^+$, we can apply $g(\cdot)$ obtaining $\mu g(\rho f(v)) \le \mu g(u) = v$.
(ii) The proof of $\rho f(v_R) < u_R$ and $S^M = \varnothing \Rightarrow S^J \ne \varnothing$ is similar.
(iii) Therefore, if one of the inequalities $\rho f(u_R) \ge v_R$, $\mu g(u_R) \ge v_R$ holds, at least one of the two sets must be empty.

(4) To construct an example it is enough to take continuous and strictly decreasing functions f and g (which can be considered as the boundaries of the sets K^J, K^M) with the following two consistent properties:

(i) $\rho f(v_R) > \rho u_R$.

(ii)

$$\exists\ (u', v') \in (\rho u_R, u_R) \times (\mu v_R, v_R) : \mu g(u') = v',\ \rho f(v') = u',$$

$$u \in [\rho u_R, u') \Rightarrow \rho f(\mu g(u)) < u \text{ while } u > u' \Rightarrow \rho f(\mu g(u)) > u.$$

By property *(ii)*, $u \geq u_R \Rightarrow \rho f(\mu g(u)) > u$, so that $S^J = \varnothing$, while if for $v \in [v_R, v+]$ we define $u = \rho f(v)$, $v \geq v_R \Rightarrow v > v' \Rightarrow u = \rho f(v) < \rho f(v') = u' \Rightarrow \rho f(\mu g(u)) < u \Rightarrow \mu g(\rho f(v)) > v$, so that $S^M = \varnothing$. On the other hand, $\rho u_R < u' = \rho f(v') < \rho f(v_R) < \rho f(v') = u' < u_R$ and similarly $\mu g(u_R) < v_R$. The other inequality in (*) follows from *(i)*.

(5) To construct an example we take continuous and strictly decreasing functions f and g with the following two consistent properties:

(i) $\rho f(v_R) > \rho u_R$.

(ii)

$$\exists\ (u_i, v_i) \in (\rho u_R, \infty) \times (\mu v_R, \infty) : \rho f(v_i) = u_i,\ \mu g(u_i) = v_i,\ i = 1, 2, 3;$$

$$\rho u_R < u_1 < u_2 < u_R < u_3;\ \mu v_R < v_3 < v_2 < v_R < v_1;$$

$$u \in [\rho u_R, u_1) \cup (u_2, u_3) \Rightarrow \rho f(\mu g(u)) > u$$

while $u \in (u_1, u_2) \cup (u_3, u^+) \Rightarrow \rho f(\mu g(u)) < u$.

The proof is similar to the proof of *(4)*.

(6) We have assumed $U^*_2 = \rho U^\#_2 = \rho u^\#_2$. We first show that for small enough $\varepsilon > 0$, $u^\#_2 - \varepsilon \geq u_R$ and $g(u^\#_2 - \varepsilon) \geq v_R$. The first of these follows by assumption, while $g(u^\#_2 - \varepsilon)$ is close to $V^\#_2 = V^*_2/\mu \geq v_R/\mu > v_R$, so the second must hold. But therefore, since $u^\#_2 - \varepsilon \notin S^J$, $\mu g(u^\#_2 - \varepsilon) \leq v^+ \Rightarrow$
$\rho f(\mu g(u^\#_2 - \varepsilon)) > u^\#_2 - \varepsilon$ Since f is continuous at V^*_2 we obtain $V^*_2 = \mu g(u^\#_2) \leq v^+ \Rightarrow \rho f(\mu g(u^\#_2)) > u^\#_2 \Rightarrow \mu g(\rho f(V^*_2)) \leq V^*_2$. It now follows that $V^*_2 \in S^M$, since (a) is true by assumption, (b) is $f(V^*_2) = f(\mu g(u^\#_2)) > (1/\rho)u^\#_2 \geq u_R$, and (c) has been proved.♦

Proof of Lemma 1

(1.1) By assumption v^*_1 exists and $v^*_1 \geq \upsilon$. Proposition 1(2) therefore implies $v^*_1 = \upsilon$ or $\mu g(\rho f(v^*_1)) = v^*_1$. However, if the later is the case, then if $u = \rho f(v^*_1) \geq \omega$, it can be verified that $u \in S^J$, which contradicts the assumption $S^J = \varnothing$.

(1.2) This is similar to *(1.1)*.

(2.1) Suppose $u_1 \leq u_2$. Then $\rho f(v_1) = u_1 \leq u^+$, so that, as $v_1 \in S^M$, $\mu g(\rho f(v_1)) \leq v_1$. Applying these inequalities,

$$v_2 = \mu g(u_2) \leq \mu g(u_1) = \mu g(\rho f(v_1)) \leq v_1.$$

The converse is proved similarly.

(2.2) Given the first condition there exists $v_1 \in S^M$ such that $u_1 = \rho f(v_1)$, so we can define $v_2 = \mu g(u_2)$, obtaining a quadruple with $u_1 > u_2$ and therefore $v_1 < v_2$, due to *(1)*, which implies the second condition. Starting from the second condition we get a similar quadruple and the converse implication. Now,

$$v^*_1 = \inf(S^M) \leq v_1 \Rightarrow \rho U^*_1 = \rho f(v^*_1) \geq \rho f(v_1) = u_1 > u_2 \geq \inf(S^J) = u^\#_2,$$

$$u^\#_2 = \inf(S^J) \leq u_2 \Rightarrow V^*_2 = \mu g(u^\#_2) \geq \mu g(u_2) = v_2 > v_1 \geq v^*_1.$$

(2.3) Each hypothesis (B) is the opposite of one of the equivalent hypothesis (A), so they are equivalent. From the hypothesis one obtains directly

$$\rho U^*_1 = \sup(\rho f(S^M)) \leq \inf(S^J) = u^\#_2,$$

$$\mu V^\#_2 = \sup(\mu g(S^J)) \leq \inf(S^M) = v^*_1.$$

(2.4) Suppose for contradiction that $U^\#_2 \geq \rho U^*_1$. As $U^\#_2 \in \Phi^J$, by Proposition 4(3) the interval $[u^\#_2, U^\#_2]$ is contained in S^J and g is constant on it. Also, by definition, $\rho U^*_1 = \rho f(v^*_1)$. Therefore,

$$\mu g(u^\#_2) = \mu g(U^\#_2) \leq \mu g(\rho U^*_1) = \mu g(\rho f(v^*_1)) \leq v^*_1.$$

Applying ρf,

$$u^\#_2 \geq \rho f(\mu g(u^\#_2)) \geq \rho f(v^*_1) = \rho U^*_1,$$

$$\rho U^*_1 = \rho f(v^*_1) \leq \rho f(\mu g(u^\#_2)) \leq u^\#_2,$$

which contradicts *(2.2)*. The proof of the second assertion is similar.

(2.5) and *(2.6)* follow from the previous results and from $U^*_2 = \rho U^\#_2$, $V^*_2 = \mu V^\#_2$.

(3) The assumptions in *(3.1)* and *(3.2)* are the meaning of consistency.

(3.1) It is clear that $\rho f(\upsilon) = \omega$ and that $v^*_1 = \upsilon$. By Proposition 1(2) $[\upsilon, v^+_1] \subseteq S^M$. But $v > v^+_1 \Rightarrow \rho f(v) \leq \rho f(v^+_1) \leq \rho f(\upsilon) = \omega \Rightarrow v \notin S^M$. Now, $\mu g(\omega) = \mu g(\rho f(\upsilon)) \leq \upsilon$. Hence either $\mu g(\omega) < \upsilon$, in which case for $u > \omega$, $\mu g(u) \leq \mu g(\omega) < \upsilon$, so $S^J = \varnothing$, or $\mu g(\omega) = \upsilon$, in which case by the argument just done for S^M, $S^J = [\omega, u^+_2]$.

(3.2) This is proved similarly.

(3.3) Using the above, $V^*_1 = \upsilon = V^*_2$ and $U^*_1 = \omega/\rho > \mu\omega = U^*_2$.♦

Proof of Proposition 5

If $v \in \Gamma^M$ and $v > \upsilon$ then $v \in S^M \cap K^J(v)$. If $v \in S^M$ Proposition 1(2) applies. If $v \in S^M \setminus K^J(v)$, then as in the proof of Proposition 1(2), conditions (a), (b) and (c) hold for $v - \delta$ if δ is small enough, but

$v - \delta \notin K^J(v)$. Hence $f(v - \delta) = f(v)$.♦

Proof of Lemma 2

(1) Neither player has a proposal the other will accept, so they both reject indefinitely.

(2) If just S^M is non-empty, (**P**, **D**) is the only strategic combination which is Nash in the games G′ and G″. It has payoffs (U^*_1, V^*_1) in G′. Consistency means

$\rho U^*_1 = \omega$ and $V^*_1 = \upsilon$. The players also do not have incentives to deviate to lower response values when beginning the game at any of the nodes 1′ and 2′, that is, when responding to any proposal, if and only if $V^*_1 \in \Gamma^M$.

(2) Only Joan can make a proposal which will be accepted by the other player. Hence (**P**, **D**) is the only strategy not leading to zero payoffs. Given Mary's response strategy, Joan cannot increment ω, nor has she any incentives to accept less at node 2′, since she can always reject and make her own proposal, which will be accepted. Mary will deviate from her response strategy $\upsilon = V^*_1$ at node $1' \Leftrightarrow \exists$ $(u, v) \in \Theta^J$ such that $\mu^2 V^*_1 \leq v < V^*_1 \Leftrightarrow \exists$ $(u, v) \in \Sigma^J$ such that $\mu^2 V^*_1 \leq v < V^*_1 \Leftrightarrow V^*_1 \in \Gamma^M$.

(3) This is similar to *(2)*.

(4) If hypothesis A holds, both players prefer their own proposal. Thus $\rho^2 U^*_1 > U^*_2 > \rho\omega$ so Joan has the incentive to increase ω so $\rho U^*_1 = \rho\omega$, while $V^*_2 > V^*_1 > \upsilon$ so Joan has the incentive to increase υ so $V^*_2 = \upsilon$. Thus hypothesis B must hold. Besides we have the results of Lemma 1(3). To solve the games G′ and G″ we have two cases: $V^*_2 \geq \mu^2 V^*_1{}'$ and $V^*_2 < \mu^2 V^*_1{}'$. The later can only occur if Φ^M is not a singleton.

(4.1) $V^*_2 \geq \mu^2 V^*_1{}'$. Mary will deviate from her response strategy $\upsilon = V^*_1$ at node $1' \Leftrightarrow \exists$ $(u, v) \in \Theta^J$ such that $V^*_2 \leq v < V^*_1$. This is impossible since $V^*_1 = V^*_2$.

(4.2) $V^*_2 < \mu^2 V^*_1{}'$. The argument on incentives to deviate is as in case *(2)*.

Proof of Theorem 3

(1) It remains to show that to each element of $\Gamma^M \cup \Xi^M$ and $\Gamma^{J\prime}$ there corresponds a perfect equilibrium. Suppose $v \in \Gamma^M \cup \Xi^M$. Set $\upsilon = v$, $\omega = \rho f(v)$. Then $S^M \neq \varnothing$, $V^*_1 = \upsilon$, and $U^*_1 = f(\upsilon)$. If $S^J = \varnothing$ we are in case (2) of Lemma 2 and $v \in \Gamma^M \setminus \Xi^M$ while if $S^J \neq \varnothing$, $v \in \Gamma^M \cap \Xi^M$ and we are in one of the cases (4.1), (4.2), according to how $V^*_1{}'$ is chosen (if there is no choice, the case is (4.1)). Since each value in $\Gamma^M \cup \Xi^M$ is different the correspondence is bijective. Suppose $u \in \Gamma^J$. Set $\omega = u/\mu$, $\upsilon = \mu g(u)$. Then $S^J \neq \varnothing$, $U^*_2 = \mu U^\#_2 = u$ and $V^*_2 = \mu V^\#_2 = \upsilon$. If $S^M = \varnothing$ we are in case (3) of Lemma 2 and $u \in \Gamma^J \setminus \Xi^J$ while otherwise we are in one of the cases (4.1), (4.2), in which it is Joan's proposal which is accepted. But these are precisely the cases we have excluded from Γ^J in the definition of $\Gamma^{J\prime}$. Thus we have a bijection between the values in $\Gamma^{J\prime}$ and the equilibria in which Mary's proposals are accepted.

(2) If Σ^J is non-empty then so is Γ^J and the game is not a principal agent game, because Mary has proposals which may be accepted. If Γ^M has more than one point then there are several equilibria, in which Mary's strategies play a role. If Σ^J is empty then Ξ^M is empty and $\Gamma^M = \{\inf(\Sigma^M)\}$, corresponding to $U^*_1 = f(v_R)$. If f is decreasing, $\Gamma^M \setminus \Xi^M$ can only contain points $\leq V^*_1$ by Proposition 5 so that is the

only element of Γ^M. There is only one equilibrium in which Joan gets the highest payoff she can get offering Mary at least her reserve utility.

(3) This situation is similar to *(2)*.♦

Proof of Theorem 4

The first thing to note is that the definitions of S^J and S^M do not depend on who begins the game. The second is that both players prefer the outcome in Table 8 to the outcome in Table 6, that is, if only Mary can make a proposal which will be accepted it is better if she makes it in period 1 than in period 2, thus avoiding the discount. In this way, for Mary, a proposal can materialize which might have been rejected by Joan if delayed to the second period because by then even the reserve utility has a discount (we keep to the convention that a proposal is accepted even when it yields only the reserve utility and that between two zero outcomes the first is preferred). The symmetrical argument holds when only Joan can make a proposal.♦

Proof of Lemma 3

(1) Condition (a) and $v \in [[v_R, v+] \cap K^J(v)$ are clear; (b) holds by the additional assumption, and (c) holds for small enough τ because: as $\tau \to 0$ $\rho f(v)$ increases to $f(v)$; therefore, since g is left continuous, $g(\rho f(v)) \to g(f(v))$ so also $\mu g(\rho f(v)) \to g(f(v)) < v$.

(2) Condition (c) cannot hold for small τ because, as in *(1)*, as $\tau \to 0$ $\mu g(\rho f(v)) \to g(f(v)) > v$.

(3) The first assertion holds by construction. f is continuous at v_0 by the left continuity of f an g.

(4) We have $u_\tau = \rho_\tau f(v_\tau)$ and $u_\tau = f(v_\tau/\mu_\tau)$. Therefore

$$\frac{(1-\rho_\tau)u_\tau}{(1-\mu_\tau)v_\tau} = \frac{\rho_\tau(f(v_\tau)-f(v_\tau/\mu_\tau))}{\mu_\tau(v_\tau/\mu_\tau - v_\tau)}.$$

We take limits of both sides as τ tends to zero. Since the numerator and denominator of the LHS tend to zero, by L'Hôpital's rule the limit is the quotient of the derivatives at zero. But

$$\frac{d}{d\tau}((1-\rho_\tau)u_\tau)\Big|_{\tau=0} = \lim_{\tau\to 0}\frac{(1-\rho_\tau)u_\tau}{\tau} = \frac{d}{d\tau}((1-\rho_\tau))\Big|_{\tau=0} \lim_{\tau\to 0} u_\tau = \alpha u$$

and similarly $\frac{d}{d\tau}((1-\mu_\tau)v_\tau)\big|_{\tau=0} = \beta v$, so we obtain

$$\frac{\alpha u}{\beta v} = - f'(v)$$

which implies the negative elasticity of f is α/β. It is clear that where f and g are inverses their elasticities are reciprocal.

(5) The limit exists because the sequence v_τ is bounded. If infinitely many of the points $v_\tau \in \Xi^M{}_\tau$, then $\mu g(\rho f(v_\tau)) = v_\tau$ so $\mu g(\rho f(v_0)) = v_0$. If infinitely many of the points $v_\tau \in \Gamma^M{}_\tau \setminus \Xi^M{}_\tau$, then they are each at a point where f is constant on some interval. If also the limit does not satisfy $\mu g(\rho f(v_0)) = v_0$ then for small enough τ, $g(f(v_\tau)) < v_\tau$. Therefore $v_\tau \in \Gamma^M{}_0 \setminus E^M{}_0$ and since this set is finite the limit is in the set.

(6) We calculate:

$$\frac{\partial h(\tau, v)}{\partial \tau} = \frac{-\beta\mu g(\rho f(v)) + \mu g'(\rho f(v))(-\alpha\rho f(v))}{v} = h(\tau, v)(-\beta + \alpha g^{\varepsilon}(\rho f(v))),$$

and integrate.

(7) Proof of (⇐). We have $g^{\varepsilon}(u) \geq \beta/\alpha$ on $(f(v_0), f(v_0-\varepsilon))$, without equality occurring on subintervals of this set. Thus $h(\tau, v) > 1$ for $\rho f(v) \in (f(v_0), f(v_0-\varepsilon))$, which for small τ is an interval ending at $f^{-1}(\rho^{-1}f(v_0)) < v_0$. Hence $\Sigma^M{}_\tau \cap (v_0-\varepsilon, f^{-1}(\rho^{-1}f(v_0))) = \varnothing$ for small enough τ.

Proof of (⇒). LHC1 is broken if on some subinterval $g^{\varepsilon}(u) = \beta/\alpha$ or if $g^{\varepsilon}(u) < \beta/\alpha$ somewhere on $(f(v_0), f(v_0-\varepsilon))$. Then the integral expression shows that for small values of τ there are fixed values
$v < v_0$ that lie in $\Sigma^M{}_\tau \cap I$.

(8) Proof of (⇐). We have $g^{\varepsilon}(u) > \beta/\alpha$ arbitrarily closely to $f(v_0)$ on $(f(v_0+\varepsilon), f(v_0))$ or $g^{\varepsilon}(u) = \beta/\alpha$ on subintervals of $(f(v_0+\varepsilon), f(v_0))$ arbitrarily closely to $f(v_0)$. Thus there are values of v arbitrarily closely to the right of v_0 for which $h(\tau, v) \leq 1$. Hence $\Sigma^M{}_\tau \cap [v_0, v_0+\varepsilon) \neq \varnothing$ for small enough τ.

Proof of (⇒). RHC1 is broken if for some smaller subinterval $(v_0, v_0+\varepsilon')$, $g^{\varepsilon}(u) \leq \beta/\alpha$ on $(f(v_0+\varepsilon'), f(v_0))$ and equality does not occur on subintervals. But then the integral expression shows $\Sigma^M{}_\tau \cap (v_0-\varepsilon\ v_0+\varepsilon') = \varnothing$ for small τ.

(9) Proof of (⇐). For ε choose the minimum ε of the relevant left and right side conditions. If $v_0 > v_R$, each of the left hand conditions implies (using *(7)* and *(8)* if necessary) that $\Sigma^M{}_\tau \cap (v_0-\varepsilon, w_\tau) = \varnothing$, where w_τ tends to v_0, while each of the right hand side conditions imply $\Sigma^M{}_\tau \cap [v_0, v_0+\varepsilon) \neq \varnothing$. It follows that the sequence $v_\tau = \inf(\Sigma^M{}_\tau \cap I)$ tends to v_0 as $\tau \to 0$. If $v_0 = v_R$, the left hand conditions are unnecessary, but LHC2 holds vacuously.

Proof of (⇒). If either of the left hand conditions is broken, then there is some $w < v_0$ for which $\Sigma^M{}_\tau \cap (v_R, w) \neq \varnothing$, while if only one of the right conditions is broken there is some ε' for which $\Sigma^M{}_\tau \cap (v_0-\varepsilon\ v_0+\varepsilon') \cap [v_R, v_0+\varepsilon') = \varnothing$ for small τ.

(10) Since $g(f(v_0)) < v_0$, for small enough τ, $\mu g(\rho f(v_0)) < v_0$. Therefore v_0 is on a segment on which f is constant (by applying Proposition 5 to $\Gamma^M{}_0$) and for small enough τ, $v_0 \in \Gamma^M{}_\tau$.

(11) Since $\Phi^M{}_\tau$ is non-empty and is contained in the bounded set P^M (Proposition 2(1)); $\Phi^M{}_0$ has at least one point.

(12) The properties $v_\tau \in [v^*{}_{1\tau}, v^+]$, $(U^*{}_{1\tau}, v_\tau) \in K^J$ and conditions (a), (b) and (c) can be verified to pass to the limit. The rest is as in the proof of Proposition 2(2).

(13) As in *(10)*, for small enough τ, $v_\tau = v^*$ so $\Phi^M{}_0 = \Phi^M{}_\tau$. If v^* is an interior point of $E^M{}_0$ then f cannot be constant in a neighborhood, so $\Phi^M{}_\tau = \{v^*{}_{1\tau}\}$. Similarly the given direct definition of $\Phi^M{}_0$ gives the singleton $\{v^*\}$.♦

Proof of Theorem 1
Lemma 3, the symmetry of the definitions of S^J and S^M (independently of which player makes the first offer) and the fact that as τ tends to zero $(U^*{}_2, V^*{}_2)$ and $(\rho U^{\#}{}_2, \mu V^{\#}{}_2)$ have the same limit show that the construction of the optimal proposals (U^{J*}, V^{J*}), (U^{M*}, V^{M*}) is correct. The proof is an application of Theorems 3, 4 and Lemma 3.♦

Proof of Theorem 2
When a player imposes a punishment, she maximizes her payoff given the other player's share. The benefit received from punishing the other is reducing the emotional discomfort due to unfairness. For this to be possible we must have $\delta > 1$. We shall assume that there are no negative punishments, and that the maximum level of punishment is 1, so $0 \le x \le 1$. When Joan makes an offer, Mary decides her punishment by maximizing

$$v(x) = s^M - x - D^M(s^J - \delta x - s^M + x)$$

given s^J and s^M. We have

$$\frac{dv}{dx} = (\delta - 1)D^{M\prime}(s^J - s^M - (\delta - 1)x) - 1.$$

There are three cases.

(i) After deciding on her punishment, Mary has more than her fair share: $s^J - s^M - (\delta - 1)x < 0$. We show this in this case $x = 0$. In this region D^M is given by $d^M{}_1$ so $D^M < 0$ and therefore $\frac{dv}{dx} < 0$, so punishment will be decreased to its minimum.

(ii) After deciding on her punishment, Mary has exactly her fair share. Then $x = (s^J - s^M)/(\delta - 1)$ and $d_1^{M\prime}(0) \le (\delta - 1)^{-1}$ because Mary has no incentive to punish more.

(iii) After deciding on her punishment, Mary has less than her fair share. Writing T_δ^M for $\min\{T, 1\}$, where T is the solution of the equation $d_1^{M\prime}(T) = (\delta - 1)^{-1}$ (the maximum level of punishment is 1), we obtain $x = (s^J - s^M - T_\delta^M)/(\delta - 1)$.

Whether case (ii) or case (iii) occurs depends only on Mary's character: how angry she gets when she is treated unfairly. By defining $T_\delta^M = 0$ in the case when $d_1^{M\prime}(0) \le (\delta - 1)^{-1}$ we can summarize these results by writing

$$x = \max\{\frac{s^J - s^M - T_\delta^M}{\delta - 1}, 0\}.$$

We now show that Joan adjusts her proposals so that Mary does not punish her, so x will be zero. Suppose $x > 0$. Then Joan's payoff is

$$u = s^J - \delta x - D^J(s^M - s^J + (\delta - 1)x) = s^J - \frac{\delta(s^J - s^M - T_\delta^M)}{\delta - 1} - d_1^J(T_\delta^M)$$

so (using $s^M = 1 - s^J$)

$$\frac{du}{ds^J} = 1 - \frac{2\delta}{\delta - 1} = - \frac{\delta + 1}{\delta - 1} < 0.$$

This means that in her proposals Joan will limit her share so as not to provoke Mary to punish her. Therefore $x = 0$ and Joan chooses her proposals from the subset

$$Q^J = \{(s^J - D^J(s^M - s^J), s^M - D^M(s^J - s^M)) : 0 \leq s^J \leq \frac{1 + T_\delta^M}{2}\}.$$

We now obtain Joan's function f. For $\frac{1}{2} \leq s^J \leq \frac{1 + T_\delta^M}{2}$, Joan's utility is $u = s^J - d_1^J(s^J - s^M)$ so $\frac{du}{ds^J} = 1 - d_1^{J\prime}(s^J - s^M)$. According to its definition, is the maximum share that Joan can take for herself without feeling such discomfort that she would prefer to have less. Then for $\frac{1}{2} \leq s^J \leq s_+^J$.Joan's utility is increasing. On the other hand, for $0 \leq s^J \leq \frac{1}{2}$, Joan's utility is $u = s^J - d_0^J(s^M - s^J)$, which is increasing with s^J since $\frac{du}{ds^J} = 1 + d_1^{J\prime}(s^J - s^M) \geq 0$. In other words, there is a maximum share s_{Max}^J that Joan can propose to take for herself, which is determined either by the restrictions imposed by Mary's punishment, or by Joan's own discomfort if she takes too much advantage of Mary, whichever happens sooner.

We analyze Mary's proposals similarly. She chooses her proposals from the set

$$Q^M = \{(s^J - D^J(s^M - s^J), s^M - D^M(s^J - s^M)) : 0 \leq s^M \leq \frac{1 + T_\delta^J}{2}\}.$$

Our analysis permits us to define the functions f and g, which we write in parametric form.

$$f(v) = s^J - D^J(s^M - s^J),\ \ v = s^M - D^M(s^J - s^M)\ \text{ given by }\ s^J \in [s_{Min}^J, s_{Max}^J],$$

$$f(v) = s_{Max}^J - D^J(s_{Min}^M - s_{Max}^J)\ \text{ for }\ v \leq s_{Min}^M - D^M(s_{Max}^J - s_{Min}^M);$$

$$g(u) = s^M - D^M(s^J - s^M),\ \ u = s^J - D^J(s^M - s^J)\ \text{ given by }\ s^M \in [s_{Min}^M, s_{Max}^M],$$

$$g(u) = s_{Max}^M - D^M(1 - 2s_{Max}^M)\ \text{ for }\ u \leq s_{Min}^J - D^J(s_{Max}^M - s_{Min}^J).$$

(See Figure 4 for an example of the graph of f and g).

Observe that f and g are inverses on the sets

$$v \in [s_{Min}^M - D^M(s_{Max}^J - s_{Min}^M), s_{Max}^M - D^M(s_{Min}^J - s_{Max}^M)],$$

$$u \in [s_{Min}^J - D^J(s_{Max}^M - s_{Min}^J), s_{Max}^J - D^J(s_{Min}^M - s_{Max}^J)];$$

f and g are constant on the closure of the sets

$$v \in [0, s_{Min}^M - D^M(s_{Max}^J - s_{Min}^M)),$$

$$u \in [0, s_{Min}^J - D^J(s_{Max}^M - s_{Min}^J));$$

and f and g are not defined outside these sets. Therefore these four sets are $E^M{}_0$, $E^J{}_0$, $F^M{}_0$ and $F^J{}_0$. The negative elasticity of f on $E^M{}_0 \setminus \{\frac{1}{2}\}$ is given by

$$f^{\varepsilon}(s^J) = -\frac{v\frac{df}{ds^J}}{f\frac{dv}{ds^J}} = \frac{(s^M - D^M(s^J - s^M))(1 + 2D^{J\prime}(s^M - s^J)}{(s^J - D^J(s^M - s^J))(1 + 2D^{M\prime}(s^J - s^M)}$$

Its derivative with respect to s^J satisfies

$$f^{\varepsilon\prime} = -\frac{\left(v'f' + vf''\right)fv' - \left(f'v' + fv''\right)vf'}{\left(fv'\right)^2} > 0,$$

since $f'' < 0$, $f' > 0$, $v'' < 0$, $v' < 0$. At $s^J = \frac{1}{2}$ the elasticity jumps across the interval $(\varepsilon_1, \varepsilon_2)$. Applying Theorem 1 we obtain the negotiation equilibrium.♦

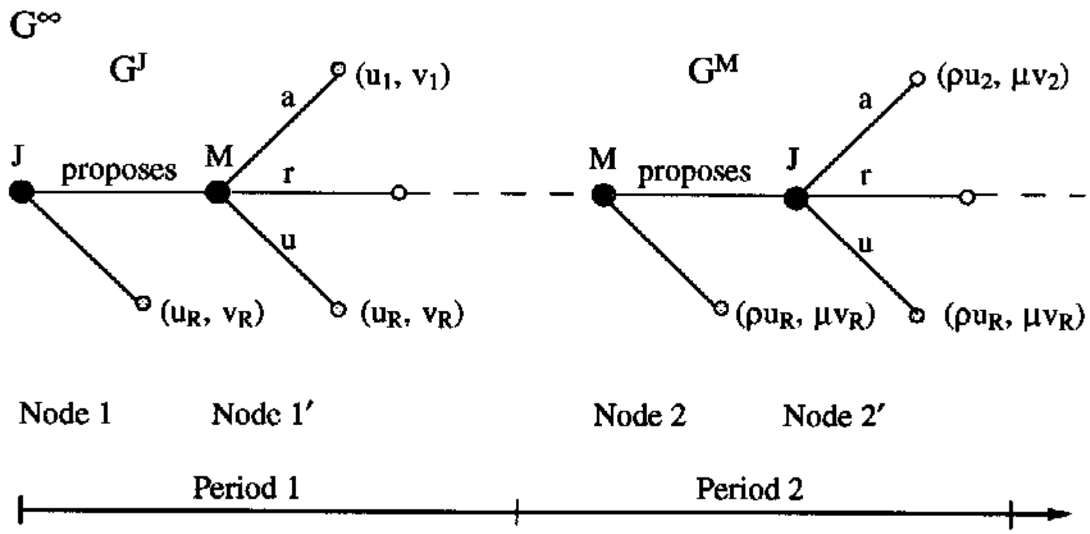

Figure 1. The game G^∞ consists of a series of basic ultimatum games G^J and G^M in which Joan or Mary make a proposal. (The payoffs are shown discounted to the first period.)

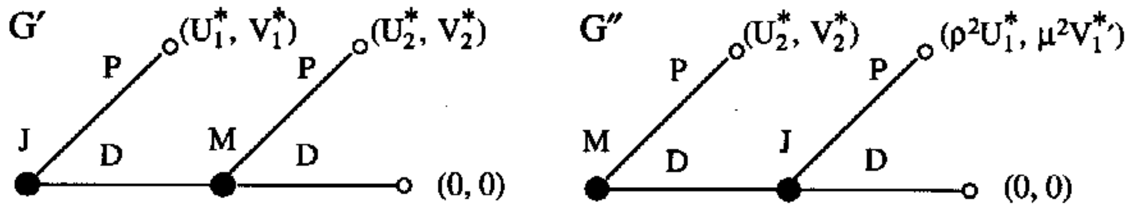

Figure 2. The games G' y G". The payoffs of these games correspond to payoffs obtained in G^∞ when playing it from node 1 or node 2 (discounted to the first period).

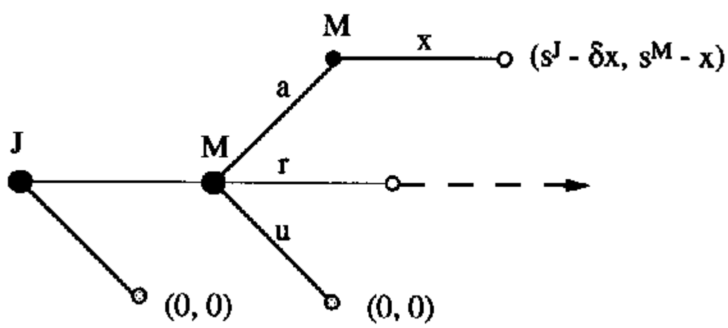

Figure 3. The one-shot ultimatum game with punishment. Joan proposes (s^J, s^M) and Mary decides on a punishment costing her x but decreasing Joan's payoff by δx. The payoffs shown are the monetary payoffs. For the resulting utilities see the text.

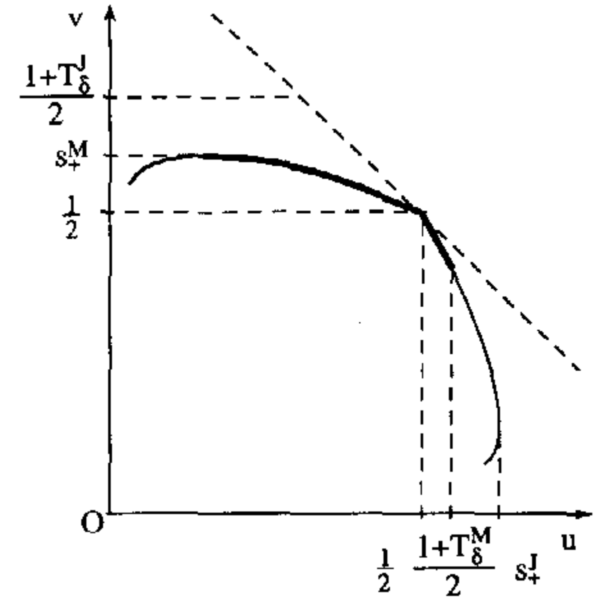

Figure 4. Negotiation when there is a value of equality. The functions f(v) and g(u), which coincide, are shown in bold on the (u, v) plane. In this case, altough both players are morally deficient, Mary's sense of fairness stops her from being unfair more than Joan's willingness to punish her, while the opposite is true for Joan.

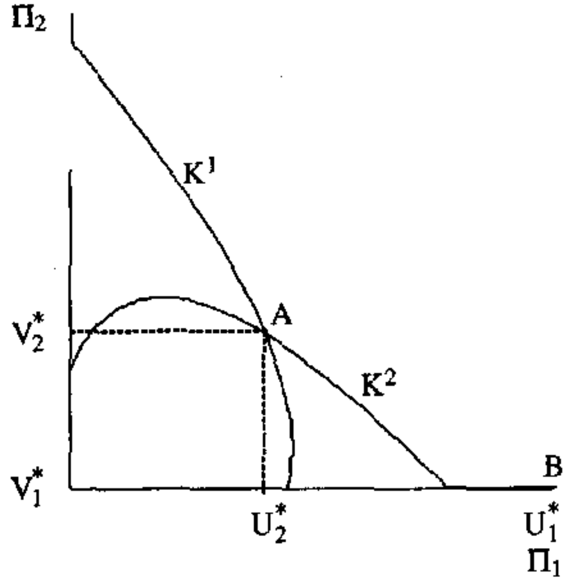

Figure 5. Duopoly Example 1. There are two perfect equilibria, A and B.

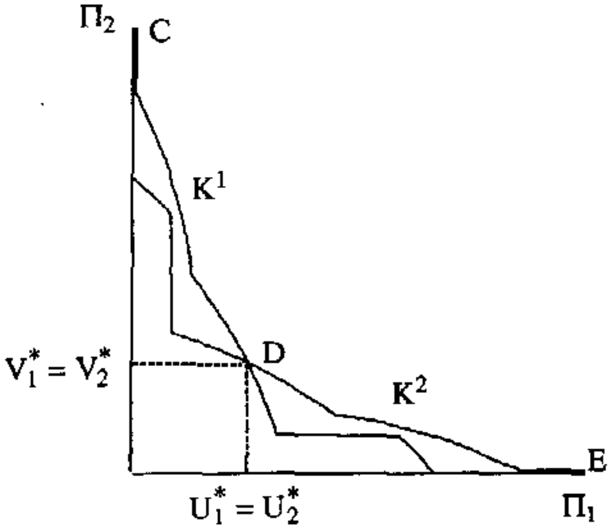

Figure 6. Duopoly Example 2. The equilibria are the large scale technology monopolies C and E. Point D does not represent an equilibrium.

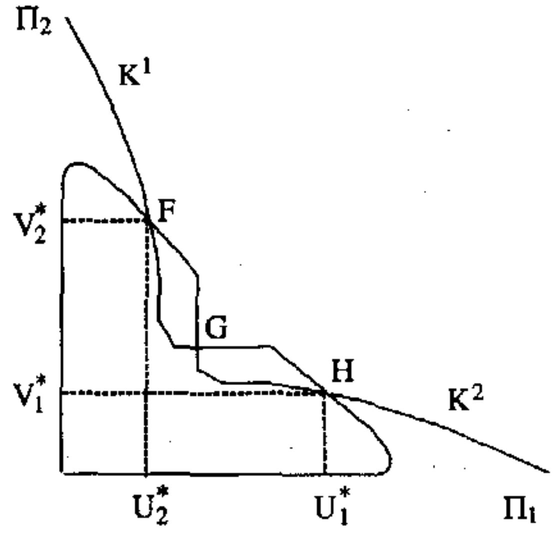

Figure 7. Duopoly Example 3. There are two equilibria F and H. In each case one firm uses the large scale technology obtaining larger profits, while the other uses the low scale technology and has lower profits. Point G does not represent an equilibrium.