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ENDOGENOUS PLANNING HORIZONS, DISTRIBUTION AND GROWTH

Introduction

Endogenous growth models havea peculiarly unsatisfactory flavor when considered from the point of view of the less developed economies. According to these models, poor countries have a higher propensity to save then rich, and the very rich could have a tendency to dissave. The classical point of view, by contrast, is that saving originates from profits, and that the poor have a lower propensity to save. This microeconomic viewpoint is consistent with the fact that there tends to be a chasm between the income of the poor and the rich. It is also consistent with the macroeconomic fact that capital tends to flow from the rich to the poor countries, and that these tend to fall into debt.

The purpose of this paper is to study distribution in an endogenous growth model in which the poor are more impatient than the rich. This will imply that *poor families do not save, while rich families do.* Such a situation may also result if the poor effectively face lower interest rates than the rich, as in Galor and Zeira [20] or Bourguignon [12]. Changes in intertemporal rates of substitution have also been considered. For example, the elasticity of marginal utility is decreasing in the Stone [33] and Geary [22] model. However these do not affect the sign of the saving rate. Increasing returns to scale can also be considered, as in Freeman [18]. Here we focus on endogenous discount rates, in which families at lower states of well-being are effectively more impatient than families who are better off, either directly for reasons of preference which are common to all agents and consequences of their state of being, or for reasons caused by these states of poverty.

The concept of time preference originates with Boehm-Bawerk [11] and Fisher [16], [17], who originates in his theory of the interest rate the idea that discount rates may depend on wealth. Formalized as the theory of recursive utility, time preferences have been analyzed by Koopmans [26], who shows by assuming limited noncomplementarity over time that welfare functions exist with variable time preference rates, and others (Beals and Koopmans [6], Uzawa [35]). It is presented extensively in Uzawa [36]. In the continuous case, a typical recursive utility functional is

$$U[c] = \int_0^\infty u(c)e^{-\int_0^t \phi(c(s))ds}dt. \tag{1}$$

The function $\phi(c)$ represents an instantaneous discount rate, so that the discount rate between two moments of time t_1, t_2 is the average of the instantaneous discount rates $\phi(c(s))$ for $t_1 \leq s \leq t_2$.

There are two strands to the theory, corresponding to the sign of ϕ'. In the first, the rich are more impatient than the poor, while in the second the reverse holds. The justification for the first position (followed by the authors mentioned above except for Fisher) is that an increase in future consumption will give more weight to present consumption (Epstein [14]). This strand of the theory is mathematically somewhat simpler since it gives rise to concave functions and therefore unique equilibria, and has been developed extensively (see for example Becker et al [8] and Becker and Boyd [9], who study optimal accumulation paths with multiple capital goods, and Joshi [24] , who introduces uncertainty). The second line, which we follow, has been studied theoretically in the context of capital accumulation and growth, for example by Mantel [30] , and finds empirical backing in Lawrance [28] . Fukao and Hamada [19] combine the two strands, studying the evolution of capital ownership under the supposition that the poor and the very rich are more impatient than those in between.

Our point of departure is that when basic needs such as food, health, childcare, housing and clothing have not been met, people are more impatient. We shall argue that (instantaneous) time preference must be considered a function of the state of well-being rather than a function of consumption. Thus, we shall consider a model in which families form their preferences over states of quality of life rather than over consumption baskets, using the utility functional (1) with consumption paths replaced by well-being paths. We shall also show that effective impatience can result not only from preferences, but from dynamic consumption effects in which present deprivation implies a future loss in utility.

We first consider a Ramsey model with preferences taken over well-being paths but with fixed intertemporal discount rates. Behavior is similar to the usual Ramsey model, except that families will not save until they reach a minimum level of well-being, and during this time their relative income worsens. However, the duration of this phenomenon is measured in years rather than decades. Then we consider a Ramsey model with endogenous time preference rates.

The model can be used to study the dynamics of distribution. We show that when the planning horizon is an increasing function of well-being, after a finite time families in an economy will fall into two classes, according to their initial level of wealth: savers and non-savers. This division will persist until wages rise enough for non-savers to begin saving, and non-savers' income will grow according to the growth of wages. We show that when (for whatever reason) the poor do not save below some level of assets, if there are two classes of technologies available for production, the first with capital and labor, and the second with capital and human capital as factors, wages may not rise, even if the economy (and savers' income) is growing. In this case a poverty trap, or income distribution trap, is possible in which, within a closed economy, the income of the poor remains constant while the income of the rich grows exponentially. If wages eventually rise and we allow investment in human capital, by mechanisms which have been widely studied income distribution may improve. Together, these two phenomena give an explanation for Kuznets' inverted U-curve.

Preferences over states of well-being

When basic needs such as food intake, health, childcare, clothing and housing have not been met, the urgency for satisfying these needs makes people more impatient. From this point of view, patience is a function of people's *state of well-being*, a term we shall use specifically in this quasi-physical sense, not to be confused with the more economic *welfare*, which refers to the benefits of income and wealth, or with *utility*, which ranks preferences. It is not a function of consumption except in so far as consumption may serve as a proxy to measure well-being.

These statements lead us to a critical analysis of the preference system stated in (1). We contend that if agents have fixed preferences at any given time (determined by the past, or by the future), their instantaneous time preference ϕ cannot be a function of consumption, because consumption is subject to the agent's decision. To write $\phi = \phi(c)$ implies that the agent decides her preferences together with her consumption! If preferences can be chosen at all, and this is not what the formulation intends to say, surely it is not by this simple mechanism. The formulation of the utility functional (1), which ranks consumption paths, implies that discount rates will be formed when consumption paths are ranked. As a consequence, for example, in a situation in which consumption is volatile, instantaneous discount rates will be volatile and result from external events such as market fluctuations. The model yields smoothly changing preferences only ex-post, as a result of consumption smoothing. It is also inconsistent to think of ϕ as depending on wealth. Such a concept would have to include capital, salaries, and other forms of wealth. But the problem would then be that choices would be taken over wealth trajectories, when wealth is but a means to welfare.

To give a sounder treatment of endogenous time preference, we develop a model in which families form their preferences over *paths of well-being*. We shall think that agents' instantaneous time preferences are a function of their state of well-being. The agents' preferences will change only once their state has changed, and not as a consequence of decisions taken at the present time. The states of well-being we are thinking of (health, nutrition, childcare, housing, clothing, and so on) can be considered as capital (part of it human), which, so far as it leads to utility, is *non-productive* in the same sense that consumption is non-productive. Since we are working in the context of a model in which a single physical good is produced, we shall consider that well-being is also a single good b (over which preferences are taken), which is produced by consuming the physical good.

Thus, our reformulation of equation (1) is

$$U[b] = \int_0^\infty u(b) e^{-\int_0^t \phi(b(s)) ds} dt, \tag{2}$$

where

$$\dot{b} = -\varkappa b - \omega + c. \tag{3}$$

In this formulation the agent chooses over paths of a state variable $b(t)$ representing her state of well-being. The role of consumption is to improve states of well-being, offsetting its natural tendency to diminish and decay. Here $\varkappa$ is the rate of decay, and ω an additional absolute rate of decrease of well-being. Although we could consider that consumption could have an additional, direct effect on utility, for example by writing $u = u(b, c)$ (but not $\phi = \phi(b, c)$), for simplicity we shall think that agents form their preferences over paths of well-being only. Thus agents will form preferences over paths of well-being, which will measure the state of a dynamical system — human beings — which tends to decay but is maintained by consumption. Notice that although preferences over streams of consumption or well-being are related, they are irreducible in that

$$u\left(b(t)\right) = u\left(b(0) + e^{-\varkappa t}\int_0^t e^{\varkappa s}(c(s) - \omega)ds\right) \tag{4}$$

cannot be expressed in terms of $u(c(t))$. When preferences are taken over states b, consumption has a durable effect.

Summing up, in our alternative formulation an agent chooses amongst feasible consumption paths c, knowing that they result in well-being paths b, which she ranks using (2), and concomitant discount rates $\phi(b)$. Shifts in consumption will change her discount rates only in the future, as a consequence of the resulting change in well-being.

Some forms of well-being are not tradeable, such as health and nutrition. Others are tradeable in that they are the direct result of maintaining a stock of goods which can be sold, such as houses, and thus have traditionally been treated as productive capital yielding utility. From our point of view, these goods can be treated in either way, as productive capital (assets which can yield a stream of consumption) or as non-productive capital (generating well-being). If assets are considered as productive capital, they enter the production function. For now, though, we shall not consider effects that well-being may have on production (e.g. health in human capital models).

To further fundament our assumption that the poor are more impatient, we shall show that there exist phenomena which can make them effectively more impatient, besides changes in preferences *per se*. These involve dynamic effects of consumption in which present deprivation results in a future loss of utility. First, the probability of being alive in the future may diminish with present deprivation. For example, Bidani and Ravallion find in a cross-country study that people with an income below US$2 per day have a life expectancy nine years shorter than those above this income level (which still includes a lot of poor). More generally, we can suppose that families maximize disability–adjusted utility, and this produces a similar effect. Although we do not model them, other phenomena can also be thought to reduce the planning horizon of the poor, such as increased uncertainty (relative to wealth), indivisibility of goods, transaction costs, etc.

Our purpose is to model behavior in which rational agents decide not to save. One of our assumptions will be a credit constraint which will mean net assets must be greater than some minimum level (such as zero). This will eliminate the unrealistic assumption that families can borrow on the basis of their full earnings into the future. First we consider Ramsey's model with preferences over b instead of c, to get an understanding of this description. This will lead to some weak non-saving results. Next, we shall introduce endogenous discount rates resulting either from preferences depending on the families' state of well-being or from dynamic well-being effects, which will effectively describe the poorer as more impatient. These imply strong non-saving results. Finally, we shall show that under these or any assumptions which imply that the poor do not save below some level of assets, unbalanced steady states exist in which the well-being of the poor is constant, while the well-being of the rich grows exponentially.

The Ramsey model in b

The Ramsey model is characterized by the optimization problem faced by the families.

$$\begin{array}{ll} & \text{Problem A} \\ \max\limits_{c(t)} & U[c] = \int_0^\infty u(c-\omega)e^{-\rho t}dt \\ & \dot{a} = (r-n)a + w - c \\ s.t. & a(0) = a_0 \\ & a \geq a_{\min} \\ & c \geq \omega \end{array} \tag{5}$$

$$\begin{array}{ll} & \text{Problem B} \\ \max\limits_{a(0),b(0),c(t)} & U[b] = \int_0^\infty u(b)e^{-\rho t}dt \\ & \dot{a} = (r-n)a + w - c \\ s.t. & \dot{b} = -\varkappa b - \omega + c \\ & a(0) + b(0) = R_0 \\ & a \geq a_{\min} \end{array} \tag{6}$$

Problem A is a slightly modified statement of Ramsey's model and Problem B is the alternative statement. a is productive capital, c is consumption, w is the wage rate. We have introduced a term ω in Problem A (following Stone [33] and Geary [22]) to force a minimum consumption level in the case when u satisfies the Inada conditions at 0, bringing it closer to Problem B, in which the term ω induces this minimum consumption naturally. n is the rate of growth of the population, which is assumed to be constant. We shall assume for simplicity that $\varkappa > n$, which is the usual case. We let $R = a + b$ represents the sum of productive and non-productive capital. The transversality condition for Problem A is $\lim_{t\to\infty} ae^{-(\bar{r}-n)t} = 0$, where $\bar{r} = \frac{1}{t}\int_0^t r(s)ds$. For Problem B we ask instead for $\lim_{t\to\infty} Re^{-(\bar{r}-n)t} = 0$ (nothing would change if we retained the original condition, but this one is more natural). A family is said to save if and only if it is not credit constrained (wishing to borrow), so $a > a_{\min}$. We shall also assume that $(r-n)a_{\min} + w - \omega > 0$, which means that a family at the credit constraint can sustain positive consumption by living on its salary. We use the usual abbreviation $\gamma = \frac{r-n-\rho}{\sigma}$.

For later comparison, we first give the solution to Problem A. All proofs (and sometimes additional results) are in the appendix.

Theorem 1 *Consider Problem A and suppose that* r *and* w *are exogenous and constant.*

Case 1: $\gamma > 0$. *The family will follow an unconstrained solution given by*

$$a = -\frac{w-\omega}{r-n} + (a_0 + \frac{w-\omega}{r-n})e^{\gamma t},\ c = \omega + (c_0 - \omega)e^{\gamma t}, \tag{7}$$

with $c_0 = (r-n-\gamma)(a_0 + \frac{w-\omega}{r-n}) + \omega$.

Case 2: $\gamma < 0$. *If* $a > a_{\min}$ *the family will dissave, following the unconstrained solution of equation (7) until* $a = a_{\min}$, *and then will follow the constrained solution*

$$a = a_{\min}, c = (r - n)a_{\min} + w. \tag{8}$$

If $a = a_{\min}$ *initially then the family will follow the constrained solution from the beginning.*

Case 3: $\gamma = 0$. *The two types of solution are identical, and will be followed by any family.*■

Solutions which are conspicuously absent in the Ramsey model are solutions in which some agents in a growing economy do not save, or do not save for a period of time and then become savers, for reasons other than those which can be attributed to variations in individual preferences.

We now turn to Problem B. For simplicity we have not used a consumption function with diminishing returns in the differential equation for b. This implies that wealth can jump between a and b, so that we are abstracting from some types of dynamics. Since no jumps occur except possibly at $t = 0$, the assumption means that the appropriate balance between a and b is attained instantly (or that the adjustment between them is fast relative to the accumulation of capital).

Theorem 2 *Consider Problem B and suppose that* r *and* w *are exogenous and constant. There are two types of solutions, according to whether the family is credit constrained or not. Unconstrained solutions (Type 1) are given by*

$$a = \frac{\varkappa + \gamma}{r - n - \gamma} b - \frac{w - \omega}{r - n}, \; b = b_0 e^{\gamma t}, \; c = (\varkappa + \gamma) b + \omega, \tag{9}$$

where $b_0 = \frac{r-n-\gamma}{\varkappa+r-n}\left[R_0 + \frac{w-\omega}{r-n}\right]$. *Constrained (Type 2) solutions are given by*

$$\begin{gathered} a = a_{\min}, \\ b = \frac{(r-n)a_{\min} + w - \omega}{\varkappa} + \left[b_0 - \frac{(r-n)a_{\min} + w - \omega}{\varkappa}\right] e^{-\varkappa t}, \\ c = (r - n)a_{\min} + w, \end{gathered} \tag{10}$$

where $b_0 = R_0 - a_{\min}$. *Define*

$$\begin{gathered} R_{\min} = \frac{\varkappa + r - n}{\varkappa + \gamma} a_{\min} + \frac{(r - n - \gamma)(w - \omega)}{(\varkappa + \gamma)(r - n)}, \\ R_{\text{eq}} = \frac{r - n + \varkappa}{\varkappa} a_{\min} + \frac{w - \omega}{\varkappa}. \end{gathered} \tag{11}$$

R_{eq} *is the equilibrium level of* R *for constrained families, while* $R_{\min}$ *is the level of* R *at which unconstrained families become constrained. We have*

$$R_{\text{eq}} > R_{\min} \Leftrightarrow \frac{\gamma((r-n)a_{\min} + w - \omega)}{\varkappa(\varkappa + \gamma)} > 0. \tag{12}$$

Suppose that $\varkappa + \gamma > 0$. Families evolve between the two types of solutions as follows.

Case 1: $\gamma > 0$. If $R_0 \geq R_{\min}$, the family will follow an unconstrained solution, while if $R_0 < R_{\min}$, the family will be initially constrained, but after a finite time will begin saving.

Case 2: $\gamma < 0$. If $R_0 > R_{\min}$, initially the family will dissave (following an unconstrained solution), eventually switching to the constrained solution leading it to $R_{\text{eq}} < R_{\min}$, while if $R_0 \leq R_{\min}$ it will be constrained from the beginning.

Case 3: $\gamma = 0$. If $R_0 \geq R_{\min}$, the family is unconstrained and has constant R, while if $R_0 < R_{\min}$, it is constrained and R tends to $R_{\text{eq}} = R_{\min}$ in infinite time.■

For us the main contrasts between Theorems 1 and 2 are the following. Firstly, the introduction of the well-being state variable b has resulted in some non-saving results. When the non-productive capital stock is too low, it must be increased before investment will occur in productive capital. In the case of economic growth, when $\gamma > 0$, modelling families as solving Problem A implies they all save, while if families solve Problem B they will begin to save only after they have reached a certain minimum level of total wealth $R_{\min}$. Secondly, in the case $\gamma = 0$, constrained families are not identical but differ in their levels of well-being, although this converges. Third, parameters such as $\varkappa$ and ω have been added naturally into consideration, affecting such quantities as the rates of convergence of the system to equilibrium.

However, we consider the non-saving results to be weak because the exponent governing the exponential convergence to saving is $\varkappa$, which originates as the exponent governing the decay of individual well-being in the differential equation for b (equation (3)). Even if we though of ω as the main term in this equation affecting the poor, $\varkappa$ would be at least the rate of depreciation of well-being when only basic needs ω are met, or, in a wider interpretation, the rate of depreciation of durable goods. In either case, the decay represented by $\varkappa$ could not have a half-time longer than a fraction of an individual's lifetime. However, our objective is to model non-saving behavior which can last through prolonged periods in the process of economic development.

These considerations force us to take into account other aspects of poverty which may induce non-saving, such as endogenous time preferences. Before doing this, we give some closed economy results relating to Problem B. Thus, we suppose that there is a production function $F(K, L)$ and that the aggregate of the family assets a equals K. The first observation is that for there to be non-saving behavior, there must be inequality of distribution, for if all families are equal, then they must own assets and the interest rate must rise to a level at which there is an incentive to save. Thus we consider the dynamics of Problem B when there is inequality of distribution. We shall suppose that the economy is divided into two sets of identical families, in one of which families are more wealthy than in the other.

Theorem 3 *Consider Problem B for a closed economy, so that r and w are endogenous. Suppose that there are two sets of families growing at the same rate n, with total* *population N, and suppose that the proportion of families in each set is* n_1 *and* n_2 *respectively* ($n_1 + n_2 = 1$). *Let* a_i, b_i, R_i $i = 1, 2$, *represent the variables corresponding to families in the first and second sets respectively, and suppose that* $R_{10} > R_{20}$ *(the first set of families has a higher initial wealth). Let* $g_i = (\varkappa + r - n)^{1/\sigma} b_i$, *and define* $a = n_1 a_1 + n_2 a_2$, *and similarly b, R and g. The total capital per capita in the economy is* $k = a$.

(1) Suppose first that all families are saving. Introducing the change of variables $g = (\varkappa + r - n)^{1/\sigma} b$, *we obtain the system of simultaneous equations*

$$\frac{\dot{g}}{g} = \gamma, \tag{13}$$

$$\begin{aligned}\left\{1 - \tfrac{1}{\sigma} f''(k)(\varkappa + r - n)^{-\frac{1+\sigma}{\sigma}} g\right\} \dot{k} = \\ f(k) - (\delta + n)k - (\varkappa + \gamma)(\varkappa + r - n)^{-1/\sigma} g - \omega.\end{aligned} \tag{14}$$

where $\gamma = \frac{r-n-\rho}{\sigma}$, $r = f'(k)$. *The loci of* $\dot{g} = 0$ *and* $\dot{k} = 0$ *in the* (k, g) *plane are given by:*

$$\begin{aligned}\dot{g} = 0 \Leftrightarrow \quad & k = k^*, \text{ where } f'(k^*) = \rho + n, \\ \dot{k} = 0 \Leftrightarrow \quad & g = \frac{\sigma\left(\varkappa + r - n\right)^{1/\sigma}\left(f(k) - (\delta + n)k - \omega\right)}{r + \sigma\varkappa - \rho - n}.\end{aligned} \tag{15}$$

The phase-diagram is of the type of the usual Ramsey diagram except for a possible asymptote for k in the case when $\sigma\varkappa < \rho + n$, *in which case k does not go beyond* $k_{\max}$ *(where* $f'(k_{\max}) = \rho + n - \sigma\varkappa$*) even when b becomes large. However, the qualitative behavior of the solutions is unaffected by this asymptote (see Figures 1a and 1b).*

(2) Suppose now that not all families save. Then only the families in the second set do not save (since there must be positive assets in the economy). The per-capita amount of capital in the economy is $k = n_1 a_1 + n_2 a_{\min}$. *Two equations describe the variables of the first set of families, equation (13) with* g_1 *instead of g, and*

$$\begin{aligned}\left(\tfrac{1}{n_1} - \tfrac{1}{\sigma} f''(k)(\varkappa + r - n)^{-(1+\sigma)/\sigma} g_1\right) \dot{k} = \\ = f(k) + \left(\tfrac{n_2}{n_1} r - \tfrac{1}{n_1} n - \delta\right) k - (\varkappa + \gamma)(\varkappa + r - n)^{-1/\sigma} g_1 - \omega.\end{aligned} \tag{16}$$

The behavior of families in set 2, who are not saving, is described by $a_2 = a_{\min}$, $\dot{b}_2 = -\varkappa b_2 + (r - n) a_{\min} + w - \omega$.■

One important feature of the solution when distribution is not equal is that while the second set of families does not save, *the relative distribution of real assets worsens*, since a_1 grows while a_2 remains at $a_{\min}$. Another feature is that distribution affects wealth in the following sense. Suppose the number of non-saving families increases while the number of saving families remains unchanged. Then there will be a higher demand for capital, so in a closed economy interests will rise and the saving families will become wealthier. Thus *families with the same initial wealth will become richer in poorer societies*.

Theorem 3 allows us to understand the nature of the solution of the system in the case of unequal distribution by using first one and then another Ramsey-type phase diagram. Near the equilibrium behavior is governed by the phase diagrams in Figures 1a or 1b. If some non-saving behavior occurs for the case of a growing economy, the growth path will first be governed by a similar phase diagram, derived using (16) instead of (14). If the economy begins with a suboptimal level of capital, a trajectory staring near the bottom left-hand corner will be chosen which reaches the corresponding trajectory in Figures 1a or 1b at a point at which $R_2 = R_{\min}$. The g_1 axis is proportional to the g axis, with the constant of proportionality also depending on the solution to the problem. This procedure does not represent a full graphical solution since the identities which make the two graphs fit together are not obtained graphically, but it does give a good qualitative idea of the solutions. The system has four equations, and detailed comparisons of distribution and rate of convergence along the trajectories would require further analysis or a numerical study.

Notice that linearization at the steady state does not capture non-saving behavior, since by the time it is reached all families are saving This is yet another weakness of the model without endogenous discount rates. Here we prefer to study a model with stronger non-saving results, which in fact will lead us to some simpler aggregate economies and models in which the linearization of the steady state does involve non-saving behavior.

Endogenous planning horizons

> *Patience – a minor form of despair disguised as virtue* – Ambrose Birce.
> *I've known what it is to be hungry, but I always went right to a restaurant* – Ring Lardner.
> *Of all the preposterous assumptions of humanity over humanity, nothing exceeds most of the criticisms made on the habits of the poor by the well-housed, well-warmed and well-fed.* - Herman Melville (1819-1891).
> *The conspicuously wealthy turn up urging the character-building value of privation for the poor* – John Kenneth Galbraith.

Following our discussion on endogenous time preferences, we shall suppose that families maximize

$$U[b] = \int_0^\infty u(b)e^{-d[b,0]}dt \quad \text{where} \quad d[b,t_0] = \exp\left[\int_{t_0}^t (\phi(b) + \rho_0)\, ds\right]. \tag{17}$$

ρ_0 is a minimum discount rate. The function $\phi(b)$ completing the instantaneous intertemporal discount rate is thought of as decreasing slowly and smoothly, when compared to changes in u. One of the properties of the functional $U[b]$ is that the problem it poses is dynamically consistent, since $d[b, t_1] = d[b, t_0] + const.$ It is clear that ϕ may differ with historical, social and cultural contexts.

However, we need not assume that preferences change with well-being to obtain a problem in which the planning horizon is endogenous and families maximize (17). We give two examples of this, but first make explicit our assumptions about u and ϕ. $u(b)$ is defined on some interval $[0, \infty)$ on which it satisfies the usual conditions $u(0) = 0$, $u \geq 0$, $u' > 0$, $u'' < 0$, and $\lim_{b\to\infty} u' = 0$ (we shall not require the remaining Inada condition $u'(0) = \infty$, but this condition may also hold). ϕ satisfies the following properties on the interval $(0, \infty)$.

$$\begin{array}{c} \phi \geq 0, \lim_{b\to\infty} \phi = 0, \phi' < 0, \lim_{b\to\infty} \phi' = 0, \\ \lim_{b\to 0} Z = \lim_{b\to\infty} Z = 0. \end{array} \tag{18}$$

where $Z = -\dfrac{\phi' u}{u'}$. We shall write $\sigma(b) = -\dfrac{bu''}{u'}, \theta(b) = -\dfrac{b\phi''}{\phi'}$. Examples functions satisfying (18) are $u = \frac{b^{1-\sigma} - b_{\min}^{1-\sigma}}{1-\sigma}$, $\phi = \rho_1 \frac{b^{1-\theta}}{\theta - 1}$ on $[b_{\min}, \infty)$, where $\theta > \max\{1, \sigma\}$ together with some appropriate extension on $[0, b_{\min}]$, where $b_{\min}$ may be chosen small enough to lie below relevant empirical values. Then

$$\lim_{b\to\infty} Z = \lim_{b\to\infty} \rho_1 b^{\sigma-\theta} \frac{b^{1-\sigma} - b_{\min}^{1-\sigma}}{1-\sigma} \to 0.$$

In the first example of an endogenous planning horizon, all families have a constant subjective time discount rate $\rho_{00} < \rho_0$, but the probability of members of each family being alive (following the Yaari [37] and Blanchard [4] finite horizon model) at time t_1 given that they are alive at time t_0 *depends on their well-being* and is given by

$$P(A_{t_1} \mid A_{t_0}) = \exp\left[-\int_{t_0}^{t_1} (\phi(b) + \rho_0 - \rho_{00})\, ds\right] = e^{-(d[b,0] - \rho_{00} t)},$$

where A_t means "alive at t". This probability satisfies the independence condition

$$P(A_{t_2} \mid A_{t_0}) = P(A_{t_2} \mid A_{t_1}) P(A_{t_1} \mid A_{t_0})$$

for any $t_0 < t_1 < t_2$. Thus

$$\int_0^\infty E(u(b)) e^{-\rho_{00} t} dt = \int_0^\infty p(t) u(b(t)) e^{-\rho_{00} t} dt = \int_0^\infty u(b) e^{-d[b,0]} dt, \tag{19}$$

where $p(t) = P(A_t \mid A_0)$, so $U[b]$ is the expected value of *utility while alive*.

The second example is a reinterpretation of the first, in which $p(t)$ now represents a weight applied to utility in terms of the weighted history of well-being $d[b, t_0]$. Each life period contributes with a factor, and the worse a state of well-being, the larger its effect. A combination of the two approaches could be interpreted as *disability adjusted utility* and would be similar to the concept of disability adjusted life years. This concept can be summarized in very different words as follows: basic needs are more urgent (so there is less patience) because when they are not satisfied the result is an impairment which diminishes future utility.

A further interpretation of the problem in which families maximize (17) is one in which we simply suppose that poverty shortens effective planning horizons, not only through considerations such as health and disability, but also through increased uncertainty (relative to wealth), indivisibility of goods, transaction costs, etc., which would have to be adequately modelled.

Lawrance [28] attempts to estimate the effect of income on time preferences in the US by using panel data. Though measurement error is a problem due to the delicate nature of these calculations, she concludes that there are considerable differences in the intertemporal preferences of rich and poor households: "...three to five percentage points (...). Controlling for race and education widens this difference (...) from 12 percent for white, college-educated families in the top 5 percent of the labor income distribution to 19 percent for non-white families without an education whose labor incomes are in the bottom fifth percentile." [ibid, abstract].

Our examples and empirical studies show that poverty shortens the horizon of rational decisions. Although it is clear that this is not the only cause or effect of poverty, we shall study the consequences that this assumption has on the dynamics of income distribution. Thus we consider the following Problem C, which is a generalization of Problem B:

$$\begin{array}{ll} \underset{a(0),b(0),c(t)}{max} & U(a(0),b(0),d(0)) = \int_0^\infty u(b)e^{-d}dt \\ s.t. & \dot{a} = (r-n)a + w - c \\ & \dot{b} = -\varkappa b - \omega + c \\ & \dot{d} = \phi(b) + \rho_0 \\ & a \geq a_{\min} \\ & a(0) + b(0) = R_0 \\ & \lim_{t\to\infty} Re^{-\bar{r}t} = 0 \\ & d(0) = d_0 \end{array} \tag{20}$$

Here $\phi(b) + \rho_0$ is the instantaneous discount rate at any time. The problem is dynamically consistent since the integral of d essentially only depends on future considerations, with the past only entering as a constant factor to which the maximization problem is invariant. Thus there is no incentive later to change any decision taken now about the future, since the problem faced later is not different to the one considered from the present time.

An important point differentiating this problem from Problems A and B is that it is not invariant to the addition of a constant to the utility function, so that the absolute level of utility matters.

Theorem 4 *Consider Problem C. There are two types of solutions, accord-ing to whether the family is credit constrained or not. Let* $G = \frac{u'e^{-d}}{(\varkappa+r-n)\mu}$. *Unconstrained (Type 1) so-*

lutions satisfy the system of differential equations

$$\frac{\dot{b}}{b} = \frac{A}{B},$$
$$\dot{G} = \left[r - n - \sigma\frac{A}{B} - \phi - \rho_0 - \frac{\dot{r}}{\varkappa + r - n} \right] G, \tag{21}$$

together with

$$\dot{R} = (r-n)R - (\varkappa + r - n)b + w - \omega. \tag{22}$$

(obtained from the equation for a *using the substitution* $a = R - b$*), where*

$$A = A(b, G, r, \dot{r}) = r - n - \frac{\dot{r}}{r + \varkappa - n} - \Psi G,$$

$$\Psi(b) = \phi(b) + \rho_0 - \frac{u\phi'}{u'},$$

$$B = B(b, G) = \sigma G + \theta(1 - G);$$

the constraints $0 \leq G \leq 1$*, and the boundary conditions*

$$R = R_0, \lim_{t\to\infty} Re^{-\bar{r}t} = 0, \lim_{t\to\infty} G = 1.$$

For the case in which r *and* w *are exogenous and constant, a phase diagram can be constructed for subsystem (21). The loci of* $\dot{b} = 0$ *and* $\dot{G} = 0$ *are given by*

$$\dot{b} = 0 \quad \Leftrightarrow \quad G(b, r) = \frac{r - n}{\Psi},$$
$$\dot{G} = 0 \quad \Leftrightarrow \quad G(b, r) = \frac{W}{W + Z}. \tag{23}$$

where $W = \theta(\phi + \rho_0) + (\sigma - \theta)(r - n)$ *(for* Z *see (18)). In the (interesting) case in which* $\phi(0) + \rho_0 > r - n > \rho_0$ *(see Figure 2) some families become savers in the long run while others do not, depending on their initial level of wealth: besides a constant solution* (b^*, G^*, R^*) *obtained from the unique unstable fixed point of system (23), there are two cases of solutions satisfying the boundary conditions. In the first* $b, R \to \infty$*,* $\frac{b}{R}$ *tends to a constant, and the growth rate tends to* $\gamma_\infty = \frac{r-n-\rho_0}{\sigma}$*, and in the second* $b, R \to 0$*,* b *tends to a growth rate* $\gamma_0 = \frac{r-n-\phi(0)-\rho_0}{\sigma}$*. Each of the solutions has the line given by*

$$R = \frac{\varkappa + r - n}{r - n - \gamma} b - \frac{w - \omega}{r - n} \tag{24}$$

as an asymptote, with γ *replaced by* γ_∞ *or* γ_0 *respectively. This shows that if* $b \to 0$*,* a *eventually becomes negative, so families will switch to the constrained solution (unless* $a_{\min} \leq -\frac{w-\omega}{r-n}$*). The functions* $R(t)$ *of Type 1 trajectories are strictly monotonic and tend either to infinity or to zero, unless they correspond to the constant solution* $R = R^*$*.* $R \to \infty$ *if and only if* $R_0 > R^*$ *and* $R \to 0$ *if and only if* $R_0 < R^*$*.*

Problem C has the same constrained solutions as Problem B. d *is obtained by integration.*■

By changing Figure 2 so that (b^*, R^*) is above the line $a = a_{\min}$, it can be seen that it in some cases it is possible for a family which is initially credit restricted to reach a point when it begins saving.

Observe also in the examples provided in Figure 2 that it can happen that families who eventually save may initially sacrifice their well-being, while families who eventually do not may initially choose higher levels of well-being, since the future will be discounted at a higher rate. This kind of behavior is only explained by endogenous preferences, and not, for example, by a dependence of interest rates on assets or by a changing elasticity of substitution σ.

The propensity of the poor to consume will be even higher once they reach the credit constraint. Non-savers, who consume their full income, will tend to spend any additional income immediately, while savers will increase b immediately but also postpone some consumption. Suppose there is a small unexpected permanent increase in wealth ΔR_0 at $t = 0$ and let $C = (n_1 c_1 + n_2 c_2)N$ be the aggregate consumption. Deriving equations (9) and (10) with respect to R_0, there will be a jump in b (attained by an instantaneous burst in consumption), and a corresponding increase in permanent consumption, so

$$\begin{aligned} \Delta C|_{t=0} &= \Delta b_0 = \left[\frac{r-n-\gamma}{\varkappa+r-n} n_1 + n_2\right] \Delta R_0, \\ \Delta C(t) &= (\varkappa+\gamma)\frac{r-n-\gamma}{\varkappa+r-n} e^{\gamma t} n_1 \Delta R_0. \end{aligned} \tag{25}$$

The aggregate marginal propensity for immediate consumption is $\frac{r-n-\gamma}{\varkappa+r-n} n_1 + n_2$, which is between zero and one. Thus for permanent changes in wealth the effect on consumption is a mixture of what the permanent income hypothesis and the Keynesian aggregate consumption function predict, even for the non-constrained families, because they increase their capital b immediately. Suppose instead there is a small transitory increase in wealth, that is, a fluctuation in income with zero net effect in wealth. This will not affect the rich, so we may represent it as Δw with $\int_0^\infty \Delta w e^{-(\bar{r}-n)t} dt = 0$. Then

$$\Delta C(t) = n_2 \Delta w(t). \tag{26}$$

Now the aggregate marginal propensity to consume is n_2 and the aggregate consumption function is Keynesian. Thus our model predicts that economies with non-savers have a Keynesian aggregate consumption function a propensity to consume proportional to their number, and that the propensity to consume is larger for permanent than for temporary changes in income, because of the increased investment in b.[1]

Another phenomenon explained by this model is that the poor do have a reserve in the sense that they may divert their income steam to emergency uses if necessary, in effect borrowing from their level of well-being (such an effect is not obtained in Problem A).

Since whether families eventually save or not depends on their initial wealth,

[1] It is interesting to mention that, using the Yaari [37] and Blanchard [4] model in a monetary economy with Keynesian unemployment, Rankin and Scalera [32] find that a positive probability of death increases the short- and long-run multipliers of government spending. This will also hold in our model, if more families become non-savers.

the theorem implies that families will divide into two sets, one in which they eventually save, and another in which they are eventually constrained. After some time has passed, the non-saving families will tend to approximately similar incomes, and will not have any earnings from capital. Once this has happened, only a raise in the wage level will bring change the non-savers into savers. This could happen, for example, when enough capital has been invested by the saving families. If this is the case, non-saving behavior will not be observed in the steady states, although it will be a feature of the transition.

Mechanisms such as those described in Banerjee and Newman [1] or Loury [29] , in which there exist stochastic phenomena which may make families richer or poorer, could convert these distributions into continuous distributions. However, endogenous intertemporal preferences may further skew the distributions or introduce more than one peak.

It may be mentioned that qualitatively similar income distribution results can be obtained in a model in which agents choose over consumption paths using the utility functional (1). Here consumption would proxy for states of well-being, and the stability of preferences would be an ex-post result of consumption smoothing rather than a clear assumption. However, the similarity would end as soon as estimation became a point of interest, and the aggregate consumption function would not react differently to permanent changes in income.

A poverty trap

Once it is admitted that families may not save below a certain level of well-being, it becomes clear that the segment of the population with low income may take a long time to begin saving. These families only save once their level of well-being reaches a certain minimum. In a growing economy the income of families whose income is mainly interest will have growth rate γ while the income of non-savers will deteriorate until it reaches the floor provided by wages and then growth at the rate that wages grow. Thus the crucial question becomes whether the saving and investment of the better-off segments of the population will result in a general increase of wages, raising the income of the poor enough for them to begin saving. Once this happens, it is possible that their relative income will improve (see the section on Kuznets' U-curve below). How long wages take to increase will thus be a factor determining qualitative changes in income distribution.

The purpose of this section is to show, however, that if two sets of technologies coexist, one which is capital intensive and requires investment on the part of the participants (e.g. in human capital), and the other based on capital and labor, it is possible that wages will not rise, so that the segment of the population who saves will see its income level grow exponentially, while the non-saving segments' income will remain constant. Let

$$F(K_1, K_2, L) = A_1 K_1 + A_2 K_2^{\alpha} L^{1-\alpha}. \tag{27}$$

F is a production function which is constituted of two technologies which are perfect substitutes for each other (we suppose that enough goods exist so that in equilibrium production with the first technology can be substituted by production with the second without any problem). The first component technology has capital and human capital as factors, while the second has labor and capital as factors. Since we have not included human capital in the family's decisions, we do not include it explicitly in F, but K_1 (and also K_2) may be though of as including physical and human capital, in which case each would have diminishing returns. For example, we may think of A_1K_1 as representing a late twentieth century modern sector, and $A_2K_2^{\alpha}L^{1-\alpha}$ as representing more traditional production and manufacturing before the human capital revolution.

To simplify the application of the model with endogenous preferences, we set

$$\phi(b)=\begin{cases}\rho_1 & b\leq \underline{b},\\ \phi_1(b) & \underline{b}\leq b\leq \overline{b},\\ 0 & \overline{b}\leq b.\end{cases}\tag{28}$$

We suppose $b_{\text{eq}}\equiv\frac{(r-n)a_{\min}+w-\omega}{\varkappa}\equiv R_{\text{eq}}-a_{\min}<\underline{b}<\overline{b}$ (see definition 11)and choose ϕ_1 so that ϕ is smooth and satisfies the appropriate conditions (18).

Theorem 5 *Consider a closed economy with production function F given by (27) in which there are two sets of families growing at the same rate n, with total population N, providing their labor inelastically. Let the proportion of families in each set is n_1 and n_2 respectively ($n_1+n_2=1$). Let $R_{20}=R_{\text{eq}}$ and suppose $R_{10}=\frac{\varkappa+r-n}{r-n-\gamma}b_{10}-\frac{w-\omega}{r-n}$ where $b_{10}\geq\overline{b}$, and R_{10} is sufficiently large for investment to occur in both forms of production. The trajectories*

$$\begin{array}{ll} a_1=\frac{\varkappa+\gamma}{r-n-\gamma}b_1-\frac{w-\omega}{r-n} & a_2=a_{\min}\\ b_1=b_{10}e^{\gamma t} & b_2=b_{20}\\ c_1=(\varkappa+\gamma)b_1+\omega & c_2=(r-n)a_{\min}+w\\ d_1=(\rho_0+\rho_1)t & d_2=\rho_0 t\end{array}\tag{29}$$

$$r=A_1,\ w=(1-\alpha)A_2^{1/(1-\alpha)}\left(\frac{\alpha}{A_1}\right)^{\alpha/(1-\alpha)},\ L=N,$$
$$K_1=\left(n_1a_1+n_2a_{\min}-\left[\frac{\alpha A_2}{A_1}\right]^{1/(1-\alpha)}\right)N,\ K_2=\left[\frac{\alpha A_2}{A_1}\right]^{1/(1-\alpha)}N.\tag{30}$$

describe the behavior of the economy.■

The main feature of the solution is that the well-being b_1 of the first set of families, who are better off, grows exponentially (and so also approximately their consumption and assets), while the well-being b_2 of the second set of families, who are worse off and do not save, remains constant.

We now show that studies of convergence, which concentrate on the capital to labor ratio, could well miss the unbalanced growth present in this model. To see this suppose that A_1K_1 is a simplified representation of the production function of developed countries, while $A_2K_2^{\alpha}L^{1-\alpha}$ is the production function of underdeveloped countries.

When a convergence study is carried out on this world economy, we can suppose that the developed countries will be found (by using an expanded model) to be close to the balanced growth path. But so also will the underdeveloped countries, where the capital to labor ratio K_2/L tends to a constant even if it is not initially at the equilibrium level. But in fact wages never rise and incomes diverge exponentially.

Even if there were some complementarity between the two technologies, this shows that it could take a long time for the lower income population to initiate saving. Moreover, the time that it takes for wages to rise is independent of the rate of convergence of other transitional dynamics, such as those underlying the optimization of the aggregate capital to labor ratio, which is what the usual notion of convergence measures. Such optimization may be attained at high and even increasing levels of disparity.

An economic Kuznets U-curve

We have shown that: 1) as long as wages remain low, if families below some level of income do not save, the population will divide into two sets of families, savers and non-savers. 2) Relative income distribution will worsen while wages grow slower than the economy. 3) that this may happen under quite plausible assumptions about technological change. Under these conditions, relative income distribution will worsen until the growth rate of wages catches up. This provides the first half of the inverted Kuznets U-curve.

Many authors consider the saving behavior of families under a credit constraint who can invest in human capital with diminishing returns, or save at the current interest rate. These families will dedicate all of their investment to human capital until its returns equal the interest rate, and will then dedicate their remaining investments to saving. Relative income distribution will improve in this situation, since for lower rates of investment there is a higher return. For articles on this topic see Loury [29] , Galor and Tsiddon (who show there may exist a Kuznets effect in this period on its own, since human capital may first concentrate on a small segment of the population, yielding high returns) [21] , Glomm and Ravikumar [23] , Barham et al [2] , Fernandez and Rogerson [15] , Mayer and Ríos [31] . These phenomena explain the second half of the Kuznets U-curve, with income distribution tending to become stationary once optimal levels of human capital have been reached.

Final remarks

In an endogenous growth model, the main determinant of saving behavior is the quantity $\gamma = \frac{r-n-\rho}{\sigma}$. If we concentrate our interest in the sign of γ, σ is irrelevant. For families identical except for their wealth to have different behaviors, at least one of the quantities r, n, ρ must depend on wealth and thus be endogenous. This paper has been concerned with the implications of planning horizons being dependent on wealth. It is also possible to consider the birth rate to depend on wealth, introducing income distribution in models with endogenous fertility (Becker et al, 1990 [7]), or to consider factors which might make the interest rate r faced by families to depend on wealth.

Preferences are in principle independent of the quantities over which decisions are taken, such as consumption and accumulation. Thus if preferences change with welfare we cannot let the instantaneous discount rate depend on consumption or on assets. However, it is not illogical to think of economic preferences as dependent on people's state of well-being. Moreover, in a dynamic model, it is consistent to consider human beings as dynamical systems in themselves, who use consumption as a means to obtain states of well-being, and who choose between such trajectories of well-being.

In this paper we first explore the consequences of this dynamic viewpoint on intertemporal decisions. If the discount rate is constant, the results are similar to the usual models, except that some degree of non-saving is explained by the families' investment in non-productive capital: nutrition, health, housing (abstracting from the productivity effects that these investments could also have, as in models in which health is considered as human capital). However, these effects decay at the same rate as well-being decays if consumption ceases, and disappear in the steady states.

If discount rates can change and the interest rate is such that the lower income groups will not save while the higher income groups will, the population divides into two classes of families, savers and non-savers, according to the initial level of wealth. The non-savers eventually have very similar asset, income and well-being levels, while the savers tend to maintain their initial distribution of wealth. For the lower income families, the possibility of saving depends on wages increasing, or on transfers from the remaining population (such as public education).

We have explained several important phenomena by introducing in an otherwise neoclassical endogenous growth context a credit constraint and the hypothesis of endogenous planning horizons (which are shorter for the poor than for the rich). Families will save only when their wealth is above a critical level, and below this level they may well wish to borrow. A functional definition of poverty arises (saving versus non–saving) which is related to the concept of marginalization, since families who do not own capital (especially human capital) have a diminished access to many institutions, especially working opportunities. As long as the wage rate is low, the population will tend to divide into two classes of families: non-savers who tend to live in approximately similar low income conditions, and savers among whom there are differences of income which tend to keep a certain proportion. The aggregate consumption function will be Keynesian and the propensity to consume is larger for permanent than for temporary changes in income, due to investments in well-being by savers when wealth increases.

During the process of economic growth relative income distribution between savers and non-savers will worsen if wages increase at a smaller rate than the economy.. This may happen if new technologies substituting older technologies require investment from the participants, or, more generally, when to participate in growth the poor must first make a competitive investment (unlike the case of unskilled labor). In these situations a poverty trap may arise in which only the income of sectors of population who are able to save can grow. The time wages take to rise may be independent of the transitional dynamics of the aggregate capital to labor ratio usually measured in studies on convergence. When we include the consideration of higher returns to low levels of some forms of investment (such as human capital), our model provides elements which help to explain Kuznets' inverted U-curve of income distribution.

Appendix

Proof of Theorem 1. We use the solutions to the usual problem, replacing c with $c+\omega$.■

Proof of Theorem 2. The Hamiltonian is:

$$H = u(b)e^{-\rho t} + \lambda((r-n)a + w - c) + \mu(-\varkappa b - \omega + c) + \eta(a - a_{\min}) \tag{31}$$

The first order conditions are:

$$\begin{aligned} 0 &= H_c = -\lambda + \mu, \\ -\dot{\lambda} &= H_a = (r-n)\lambda + \eta, \\ -\dot{\mu} &= H_b = u' e^{-\rho t} - \varkappa \mu. \end{aligned} \tag{32}$$

Observe that the Hamiltonian is linear in c. This has the implication that the variables a and b may jump, keeping R constant. Since the Hamiltonian is concave in the state variables, the jumps may only occur at $t = 0$ (Kamien and Schwartz [25, part II, section 18]). Observe also that maximizing $U(a(0), b(0))$, subject to the restriction $a(0) + b(0) = R_0$ will lead to $\frac{\partial U}{\partial a(0)} = \frac{\partial U}{\partial b(0)}$, i.e. to $\lambda = \mu$ at $t = 0$. Thus the problem is well posed.

There are two types of solutions, corresponding to $\eta = 0$ and $\eta > 0$.

Type 1 solutions: $\eta = 0$. In this case $\lambda = \mu = \mu_0 e^{-(\bar{r}-n)t}$. Differentiating the remaining equation logarithmically,

$$\frac{\dot{b}}{b} = \gamma - \frac{\dot{r}}{\sigma(\varkappa + r - n)}. \tag{33}$$

In the case when r and w are exogenous and constant, $b = b_0 e^{\gamma t}$. Using equation (22) for R, we can solve to obtain (24) and therefore (9) and the expression for b_0 in terms of R_0. c is obtained from the differential equation for b. When $a = a_{\min}$, the level of R is $R_{\min}$.

Type 2 solutions: $\eta > 0$. In this case $a = a_{\min}$. Hence $c = (r-n)a_{\min} + w$, and $\dot{b} = -\varkappa b + (r-n)a_{\min} + w - \omega$, so we obtain (10). b tends to an equilibrium level $b_{\text{eq}} = \frac{(r-n)a_{\min} + w - \omega}{\varkappa}$, so also R tends to an equilibrium level, given by R_{eq}.

Combinations of the two types of solutions when r and w are constant. The solutions are continuous in R.

Case 1: $\gamma > 0$. In the Type 1 (unrestricted) solutions, a and b are increasing, so these cannot reach a point where a becomes constrained. However, if $R_0 < R_{\min}$, initially the family will follow a Type 2 solution, but since $R_{\text{eq}} > R_{\min}$, in finite time it will switch to a Type 1 solution.

Case 2: $\gamma < 0$. The Type 1 (unrestricted) solutions of a and b are decreasing. Hence if $R_0 > R_{\min}$ the family will begin with a Type 1 solution but eventually switch to a Type 2 solution leading it to $R_{\text{eq}} < R_{\min}$, while if $R_0 \leq R_{\min}$ it will follow the Type 2 solution from the beginning.

Case 3: $\gamma = 0$. Type 1 solutions have constant R, so are followed indefinitely if $R_0 \geq R_{\min}$. Otherwise a Type 2 solution in which R tends to $R_{\text{eq}} = R_{\min}$ in infinite time occurs.■

Proof of Theorem 3. There will be no borrowing (since $a_{\min} \leq 0$), so $a = k$. The solutions will only be of Type 1, because $k > 0$. By equation (33) g has equation $\frac{\dot{g}}{g} = \frac{\dot{b}}{b} + \frac{\dot{r}}{\sigma(\varkappa+r-n)} = \gamma$ and therefore (13). Using $f(k) = rk + w + \delta k$, equation (22) can be stated as $\dot{b} + \dot{k} = f(k) - (\delta + n)k - \varkappa b - \omega$. Hence

$$\dot{k} = f(k) - (\delta + n)k - \varkappa b - \omega - \left(\gamma - \frac{f''(k)\dot{k}}{\sigma(\varkappa + r - n)}\right)(\varkappa + r - n)^{-1/\sigma} g,$$

from where equation (14) is obtained. From these equations the loci of $\dot{g} = 0$ and $\dot{k} = 0$ are obtained.

We now turn to the case with unequal distribution. g_1 satisfies equation (13). As long as families in the second set do not save, $f(k) = rk + w + \delta k = \frac{N_1}{N_1+N_2} r a_1 + w$. Hence

$$\begin{aligned} \tfrac{N_1+N_2}{N_1}\dot{k} + \dot{b}_1 &= \dot{a}_1 + \dot{b}_1 = (r-n)a_1 - \varkappa b_1 + w - \omega \\ &= \tfrac{N_1+N_2}{N_1}(f(k) - (\delta+n)k) - \tfrac{N_2}{N_1}w - \varkappa b_1 - \omega. \end{aligned}$$

Substituting the equation for $\dot{b}_1$,

$$\begin{aligned} \tfrac{N_1+N_2}{N_1}\dot{k} = {} & \tfrac{N_1+N_2}{N_1}(f(k) - (\delta+n)k) - \tfrac{N_2}{N_1}(f(k) - rk) \\ & - \varkappa b_1 - \omega - \left(\gamma - \frac{f''(k)\dot{k}}{\sigma(\varkappa + r - n)}\right)(\varkappa + r - n)^{-1/\sigma} g_1. \end{aligned}$$

Thus we obtain (16).

Once both sets of families save, g_i satisfy equation (13), $k = \sum_1^2 \frac{N_i}{N_1+N_2} a_i$, and we obtain in a similar way

$$\dot{k} = f(k) - (\delta + n)k - \omega - \sum_{i=1}^{2} \tfrac{N_i}{N_1+N_2}\left(\gamma + \varkappa - \frac{\dot{r}}{\sigma(\varkappa + r - n)}\right) b_i.$$

Writing $g = \frac{1}{N_1+N_2}\sum_{i=1}^{2} N_i g_i$, we obtain the same aggregate system as when distribution is equal.■

Proof of Theorem 4. The maximization problem is bounded by the one obtained replacing ϕ with 0. Thus (see [13]) a solution exists and it satisfies the usual first-order conditions. The problem is not convex because of the term e^{-d}, so the solutions to the first-order conditions need not be unique. We define the Hamiltonian

$$H = ue^{-d} + \lambda((r-n)a + w - c) + \mu(-\varkappa b - \omega + c) + \nu(\phi + \rho_0) + \eta(a - a_{\min}).$$

The first-order conditions are:

$$\begin{aligned} 0 &= H_c = -\lambda + \mu, \\ -\dot{\lambda} &= H_a = (r-n)\lambda + \eta, \\ -\dot{\mu} &= H_b = u'e^{-d} - \varkappa\mu + \phi'\nu, \\ -\dot{\nu} &= H_d = -ue^{-d}. \end{aligned} \tag{34}$$

Observe that $U(R_0, d_0) = e^{-d_0}U(R_0, 0)$, so the maximization process is invariant under changes to d_0. Differentiating with respect to d_0,

$$\nu(0) = -e^{-d_0}U(R_0, d_0) = -U(R_0, d_0).$$

From the differential equation for ν, writing $\nu(\infty) = \lim_{t\to\infty}\nu$,

$$\nu(\infty) - \nu(0) = \textstyle\int_0^\infty ue^{-d}dt = U(R_0, d_0) = -\nu(0).$$

Therefore $\nu(\infty) = 0$ and $\nu(t) < 0$ for all t. (We prefer to keep this unusual sign for ν and retain the usual notation e^{-d} rather than e^d with d tending to $-\infty$ for discounts.) As before, the condition $\lambda = \mu$ holds initially since we maximize in $a(0)$, $b(0)$ subject to the restriction $a(0) + b(0) = R_0$. We now solve the first-order conditions. There are two types of solutions, according to whether the agent is constrained or not.

Type 1 solutions: $\eta = 0$. In this case $\lambda = \mu = \mu_0 e^{-(\bar{r}-n)t}$. Therefore

$$(\varkappa + r - n)\mu = u'e^{-d} + \phi'\nu. \tag{35}$$

Since each of the terms in (35) is positive, $0 \le G \le 1$. Moreover, since $\lim_{t\to\infty}\phi'\nu = 0$, the solution satisfies $\lim_{t\to\infty} G = 1$. Differentiating (35),

$$\frac{d}{dt}((\varkappa + r - n)\mu) = u''\dot{b}e^{-d} - u'e^{-d}\dot{d} + \phi''\dot{b}\nu + \phi'\dot{\nu}. \tag{36}$$

Dividing (36) by $(\varkappa + r - n)\mu$ and substituting $\frac{\nu}{(\varkappa+r-n)\mu} = \frac{1-G}{\phi'}$ and $\dot{\nu}$ (see (34)),

$$\frac{\dot{r}}{\varkappa + r - n} - (r-n) = \left[\frac{bu''}{u'}G + \frac{b\phi''}{\phi'}(1-G)\right]\frac{\dot{b}}{b} + \left[\frac{u\phi'}{u'} - \phi(b) - \rho_0\right]G. \tag{37}$$

Thus we obtain the first equation in (21). The second is obtained straightforwardly by differentiating the definition for G. Together they are equivalent to the original first order conditions, except that they may admit additional solutions not satisfying the stated constraints for G.

Consider the case in which r and w are exogenous and constant. We construct the (b, G) quadrants of the phase diagram 3 of the system (21). It is clear that $B > 0$, and that the conditions for ϕ imply $\Psi > 0$, $\Psi(0) > \lim_{b\to\infty}\Psi = \rho_0$. If we suppose (beyond the statement of the theorem) that $\theta > \sigma$ then also

$$\Psi' = \phi' - \frac{u'(u'\phi' + u\phi'') - u\phi' u''}{u'^2} = (\sigma - \theta)\frac{u\phi'}{bu'} < 0.$$

The loci (23) are easily derived. The conditions given for $\phi(0)$ and ρ_0 imply that for $b = 0$, $0 < \dfrac{r-n}{\Psi} < 1$, while as $b \to \infty$, $\dfrac{r-n}{\Psi}$ exceeds 1, justifying the graph of $\dot{b} = 0$ (which is strictly increasing in the case $\theta > \sigma$). The locus $\dot{G} = 0$ has the form $G = \frac{W}{W+Z}$, with $W = \theta(\phi+\rho_0) - (\theta - \sigma)(r-n)$, $Z = -\sigma\frac{u\phi'}{u'}$. Since $Z > 0$, the graph stays in $0 \leq G \leq 1$ when W, which is monotonically decreasing, is non-negative. If instead there some finite value at which $W = 0$, we obtain a graph as in Figure 2. If instead W is bounded away from zero as $b \to \infty$ (this happens if $\frac{\theta-\sigma}{\theta}(r-n) < \rho_0$), $\frac{W}{W+Z} \to 1$ because $Z \to 0$ as $b \to \infty$ (this is one of the assumptions on ϕ). The phase diagram would now be similar to Figure 2 but except that the bottom right part of the locus of $\dot{G} = 0$ would tend to 1 instead of zero (below the equilibrium trajectory). Observe further that the two loci intersect only at $b = b^*$ given by $\phi(b^*) = r - n - \rho_0$, corresponding to a unique $G = G^*$. Since the curves intersect only once, the general form of both diagrams is fully determined. For $b < b^*$, $\dot{G} < 0$ along the $\dot{b} = 0$ curve, and viceversa. This defines the sign of $\dot{G}$ in the corresponding regions, while the signs for $\dot{b} < 0$ ($\dot{b} > 0$) above (below) the $\dot{b} = 0$ curve are easy to determine.

Observe now that when $G = 1$, $\dot{G} = Z$, which is non-zero except as $b \to \infty$ and at $b = 0$. Therefore solutions satisfying the transversality condition $G \to 1$ as $t \to \infty$, must satisfy $b \to \infty$ or $b \to 0$. See the phase diagram in Figures 2, which depicts solutions on the (b, G) plane satisfying this property.

There is a value R^*, corresponding to b^* and G^*, at which $\dot{R} = 0$, given by $R^* = \frac{\varkappa+r-n}{r-n}b^* - (w - \omega)$ which completes the unstable stationary solution. To confirm the instability of (b^*, G^*) observe the linearization of the system at this point is

$$\begin{pmatrix} \dot{b} \\ \dot{G} \end{pmatrix} = \begin{pmatrix} -\dfrac{b\Psi' G}{B} & -\dfrac{b\Psi}{B} \\ \sigma\dfrac{\Psi' G}{B} - \phi' & \sigma\dfrac{\Psi}{B} \end{pmatrix}_{(b^*, G^*)} \begin{pmatrix} b - b^* \\ G - G^* \end{pmatrix}. \tag{38}$$

The determinant and trace are positive, so the stationary point is unstable.

The remaining equation is (22). As $G \to 1$, $\frac{\dot{b}}{b} = \frac{r-n-\phi-\rho_0-Z}{\sigma}$, where the integral $\int_t^\infty Z(b(s))ds$ tends to zero since $\dot{G} = ZG$. Hence b tends to grow at rate γ_0 if it tends to zero and at a rate γ_∞ if it tends to infinity, and R can be estimated to lie between two solutions of the type of equation (24) (with b replaced by an expression growing at a fixed exponential rate) so $R \to \infty$, $\frac{b}{R}$ tends to a constant, and their growth rate tends to γ_0. Thus the straight lines given by equation (24) with γ replaced by γ_0 or γ_∞ are asymptotes of the solutions.

Consider optimal trajectories $R(t)$. These cannot first decrease (or stay equal) and then increase. This is because, by eliminating all subintervals on which R first decreases (or stays equal) and then attains the same value, we obtain a new control trajectory, with possible jumps in a, b, for which we have an increasing function $R^1(t) > R(t)$ which dominates and therefore affords a higher well-being state b then $R(t)$. Thus, trajectories $R(t)$ must eventually either strictly increase (and therefore tend to infinity) or decrease. A path $R(t)$ cannot either first increase and then decrease. If it did, there would be values t_1, t_2, for which $R(t_1) = R(t_2)$, $R(t) > R(t_2)$ for $t_1 < t < t_2$ and R is decreasing after t_2 (using the first part of this paragraph). By replacing the controls after t_2 with an indefinite repetition of the controls used in (t_1, t_2), we obtain a new control trajectory, with possible jumps in a, b, which, for which total wealth follows a trajectory $R^1(t) > R(t)$ for $t > t_2$, which must therefore increase the family's utility. Hence $R(t)$ is monotonic. It must be strictly monotonic or constant, because if it is constant in an interval then b is constant by equation (22), so $(b, G, R) = (b^*, G^*, R^*)$. But this solution is unstable and cannot be reached unless it holds for all time. Further, the increasing solutions $R(t)$ must tend to infinity, and the decreasing solutions to 0, since there are no other equilibria. Observe now that two solutions $R_1(t)$, $R_2(t)$, cannot cross. this is because then we could choose a control giving $R(t) = \max\{R_1(t), R_2(t)\}$, which would improve one of the two solutions. Therefore if $R_0 > R^* > 0$, the corresponding solution $R(t)$ must tend to infinity (it cannot tend to 0 without crossing the constant solution), and if $R_0 < R^*$ the corresponding solution $R(t)$ must tend to zero. If there is some R_0 for which R tends to zero, then $R^* > 0$. This is because for sufficiently large R_0, R tends to infinity, so there must be some value $R_1 > R_0$ above which solutions tend to infinity and below which solutions tends to zero. But, observing the phase diagram, since any solution essentially retraces one of the two solution branches, $R_1 = R^*$.■

Proof of Theorem 5. The functions a_i, b_i, c_i, d_i, $i = 1, 2$, solve optimization Problem C, by Theorem 2, since along these trajectories the instantaneous discount rates $\phi(b_i) + \rho_0$ are constant. Since returns on capital must be equal, we have $r = F_{K_1} = F_{K_2}$, so $r = A_1 = \alpha A_2 (L/K_2)^{1-\alpha}$. From here we obtain r and $\frac{K_2}{L}$ and therefore K_2. w is obtained from $w = F_L = (1-\alpha) A_2 (K_2/L)^\alpha$ and the expression for K_2/L. Finally, since $K_1 + K_2 = (n_1 a_1 + n_2 a_{\min}) N$, we can now obtain K_1, which is positive if

$$b_{10} > \frac{r - n - \gamma}{\varkappa + \gamma} \left(\frac{1}{n_1} \left[\frac{\alpha A_2}{A_1} \right]^{1/(1-\alpha)} + \frac{w - \omega}{r - n} - \frac{n_2}{n_1} a_{\min} \right) \tag{39}$$

and thus for large enough R_{10}.■

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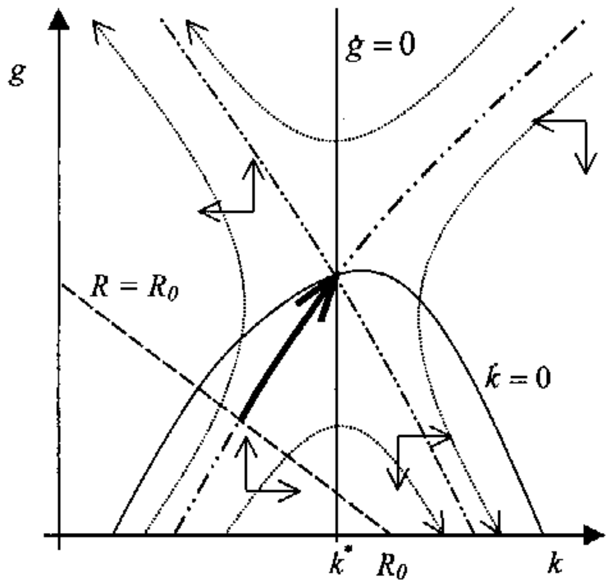

Figure 1a. Phase diagram for k, g, when $\dot{k} = 0$ has no asymptote.

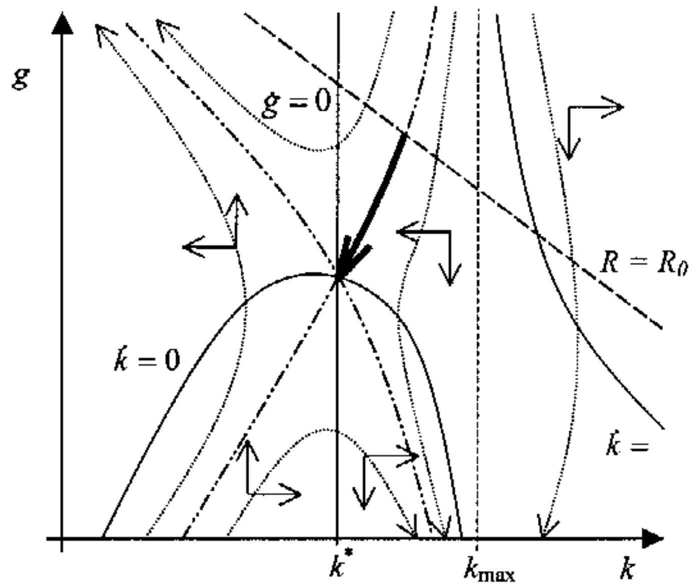

Figure 1b. Phase diagram for k, g, when $\dot{k} = 0$ has an asymptote at $k_{\max}$.

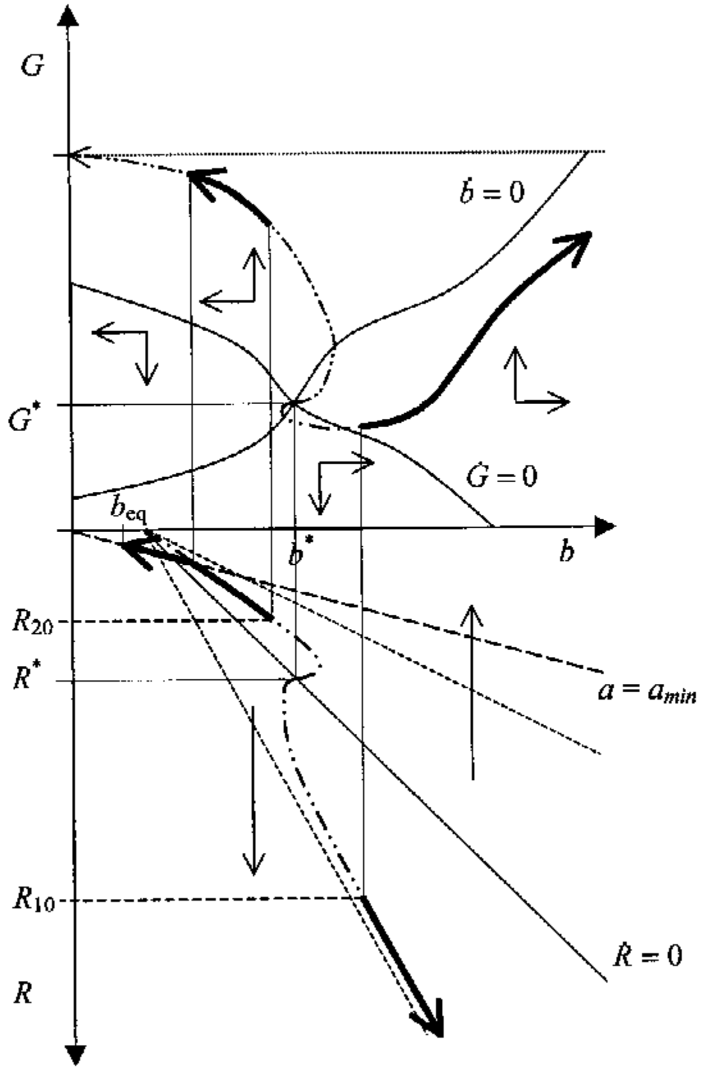

Figure 2. Dynamics of b, G, R when subjective discount rates are endogenous (W becomes zero in finite time).