CIDE

NÚMERO 95

Adolfo García de la Sienra

GAME THEORY

0. INTRODUCTION

Game theory is one of the most challenging fields for a structuralist project aimed at providing a complete reconstruction of a scientific theory.[1] There are literally hundreds –if not thousands– of papers and books in an ever-growing literature. The amount of special games continuously appearing defies any classification and makes the field to look hopelessly complicated. This makes the philosophical effort of searching for a unified field theory particularly urgent and interesting. Indeed, the question is whether all the constructs that are presented as games really constitute a theoretical *corpus* that can be reconstructed as a theory-net.

One clue in the jungle of games crowding the literature is to be found in the classic seminal work of John von Neumann and Oskar Morgenstern (VNM) *Theory of Games and Economic Behavior*. This book provides an axiomatic formulation of (finite) *n*-person games in extensive form which is both intuitive and logically rigorous. Very much in the style of the structuralist formulation of theories, VNM defined '*n*-person game' as a set-theoretic predicate. The central notion in the definition of this predicate turns out to be that of information set. The peculiar character of extensive games is therefore given by this notion and not –as many believe– by that of a tree. As a matter of fact, VNM barely make use of trees, and only for the purpose of "graphical representation".[2] Yet, in spite of the facts that VNM themselves referred to this set-theoretic formulation as "the extensive form",[3] and that their axiomatization has a straightforward game-theoretic interpretation (explicitly provided by the authors), it is no longer usual nowadays to define 'extensive game' in the way VNM did. Many authors in the field of game theory today define the concept of extensive game in terms of trees characterized by means of directed graphs.[4] There is nothing to object to this representation except that, in our view, it obscures the logical structure of the theory and makes it hard to think of extensive games which are not of a very limited kind, namely other than those in which the number of possible plays is finite.

A question that naturally arises is whether the natural, intuitive way of formulating a game provided by VNM can be extended to cover all kinds of games. This would be a method to tackle the philosophical problem of a unified field theory pointed out above. J.C.C. McKinsey raised already in 1952 the conceptual problem of extending the notion of game in extensive form to games in which function spaces are involved. Aumann (1961, 1964) provided a solution to this problem, but the solution provided here is far more general than that provided by Aumann. The aim of the present document is to provide a relatively general definition of the notion of game in set-theoretic form, from the vantage viewpoint of structuralistic philosophy of science,

[1] For a presentation of the structuralist metatheory –which cannot be given here– the reader is referred to Balzer, Moulines and Sneed (1987).

[2] Cf. VNM (1944), pp. 77-79.

[3] Cf. VNM (1944), p. 85.

[4] Since the work of Kuhn (1953). See, for instance, Bonnano (1993). Bonnano (1991) actually distinguishes games in set-theoretic form from games in extensive form.

in consonance with the state of the art in the foundations of probability theory. I leave out of consideration, however, games of infinite length as well as games with an infinite number of players. More than trying to reach the proof of a particular theorem (like, say, Kuhn's theorem on optimal behavior strategies), the aim of the document is to show the explicit logical structure of the theory. Its unifying power could be tested afterwards in terms of its yielding a criterion to distinguish the most general theorems from those that are relative to specializations of the theory.

1. THEORETICITY

The present reconstruction is based on the conjecture that the only properly theoretical terms of game theory are the concepts of utility and global probability. The reasons that support this conjecture are basically two. In the first place, people are "playing games" all of the time, but the claim of game theory is far stronger than this. The claim seems to be – intuitively – that when people play certain kinds of games they do it *in order to maximize a certain expected utility function*. I take this to be, indeed, the interesting and deep claim that makes game theory to be a non-trivial empirical theory. In the second place, the determination of particular expected utility functions – especially in economics – seems to pressupose the very claim that the theory makes. This seemingly circular procedure gives rise to Samuelson's notion of revealed preference, a notion that has been criticized by those who do not understand the role of theoretical terms in scientific theories. From a structuralist viewpoint, however, the notion of revealed preference is natural and suggests that preference – or rather utility – is a theoretical term in game theory. I hope that the discussions below will make plausible this conjecture.

2. THE PARTIAL POTENTIAL MODELS OF THE THEORY

An n-person game is seen as set of rules that allow a certain set of possible plays. When a game is actually played, a certain sequence of moves takes place. At each of these moves one of the players, or an ideal player called the umpire, must make a choice among various alternatives under conditions prescribed by the rules of the game. The umpire chooses an alternative at random, while the other players have to make a personal choice. The crucial thing in the making of the choice for these players is what information they possess about the choices that were made in the previous moves. In particular, every player must know whether it is his turn to play at the current move and the choices that are available to him at that point.

For the sake of the understanding, let us say that whoever "sees" a particular position in a play $\pi = \sigma_1, \ldots, \sigma_\nu$, for instance σ_κ ($\kappa = 1, \ldots, \nu$), can "read" in this element (which is a choice made by some player at move $\mathcal{M}_\kappa$), at least, who was the player that made such choice (i.e. who is the player whose turn was at $\mathcal{M}_\kappa$), and which were the choices available to the player at $\mathcal{M}_\kappa$. Hence, whoever "sees" all the positions in π has perfect information concerning the play. Thus, at a certain move $\mathcal{M}_\kappa$ of the game, the restriction of information for a player k consists of his not "seeing" some of the previous elements of $\sigma_1, \ldots, \sigma_{\kappa-1}$. This restriction could thus be represented by means of dashes in those places of the sequence $\sigma_1, \ldots, \sigma_{\kappa-1}$ corresponding to positions not "seen" by k, i.e. sequences of the form: $\sigma_1, \ldots, -, \ldots \sigma_{\kappa-1}$. But another way of representing this restriction is to consider all the sequences obtained from this one by substituting all the σ's that could possibly have occurred at those positions in the sequence where a dash occurs. Sets of this kind are called "information sets". For instance, an information set at move $\mathcal{M}_3$ for a little game with four moves is $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\ \sigma_1, \sigma_2', \sigma_3, \sigma_4'\}$. A player having this information set at the move does not know what happened at move $\mathcal{M}_2$, but only that it his turn at $\mathcal{M}_3$, that at $\mathcal{M}_2$ the position was σ_2 or σ_2', and that he has two possible choices, namely σ_4 and σ_4'.[5]

Therefore, the crucial game-theoretic notion of information set can be represented by means of the set-theoretic notion of partition. A *partition in* Ω is a family $\mathcal{A}$ of nonempty subsets of Ω —a subset of the power set $\mathrm{Po}(\Omega)$— such that the elements of $\mathcal{A}$ are pairwise disjoint. In the particular case in which the union of all elements of $\mathcal{A}$ happens to be Ω itself, $\mathcal{A}$ is said to be also a *partition of* Ω. But in the general case such union will be a subset of Ω. A *subpartition* $\mathcal{A}$ of $\mathcal{B}$ is a partition in Ω such that every element in $\mathcal{A}$ is a subset of an element in $\mathcal{B}$. I say that $\mathcal{A}$ is a *subpartition* of $\mathcal{B}$ *within C* iff every $A \in \mathcal{A}$ which is a subset of C is also a subset of some $B \in \mathcal{B}$ which is also a subset of C. If $\mathcal{A}$ and $\mathcal{B}$ are partitions in Ω, the *superposition* of $\mathcal{A}$ and $\mathcal{B}$ is the family of all sets of the form $A \cap B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

In what follows, the value of a function f for an argument x, $f(x)$, sometimes will be written as f_x. The terms $\mathbf{n}$ and $\boldsymbol{\nu}$, respectively, will denote the sets $\{1, \ldots, n\}$ and $\{1, \ldots, \nu\}$. The primitive terms of our axiomatization are given in the first definition, the definition of the partial models of the theory.

DEFINITION 1: $\langle \nu, n, \Omega, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \phi, \psi, \tau, \mathfrak{F}, P, d \rangle \in \mathbf{M}_{pp}(\mathbf{GAME})$ iff

(1) Both ν and n are positive integers.

(2) Ω is a nonempty set.

(3) $\mathcal{A}: \boldsymbol{\nu} \cup \{\nu + 1\} \to \mathrm{Po}(\Omega)$ is a function such that $\mathcal{A}_\kappa$ is a partition in Ω for every $\kappa \in \boldsymbol{\nu} \cup \{\nu + 1\}$.

(4) $\mathcal{B}: \boldsymbol{\nu} \to \mathrm{Po}(\Omega)$ is a function such that $\mathcal{B}_\kappa$ is a sequence of sets $(B_\kappa(k))$ $(k \in \{0\} \cup \mathbf{n})$ for every $\kappa \in \boldsymbol{\nu}$. The sets in the sequence are a partition in Ω.

(5) $\mathcal{C}: \boldsymbol{\nu} \times (\{0\} \cup \mathbf{n}) \to \mathrm{Po}(\Omega)$ is a function such that $\mathcal{C}_\kappa(k)$ is a partition in $B_\kappa(k)$.

[5] Kuhn (1953) introduced a more general definition of information set. According to this more general definition, the player does not have to know the *number* of moves that have taken place in the game, but only that it is his turn and the choices open to him. Thus, his information set looks rather like $\{\sigma_4, \sigma_4'\}$. This more general concept can be obtained out of the previous one by means of the coordinate projection operation, and so there is no loss of generality if I restrict myself, for the time being, to consider only the former.

(6) $\mathfrak{D}: \boldsymbol{\nu} \times (\{0\} \cup \mathbf{n}) \to \mathrm{Po}(\Omega)$ is a function such that $\mathfrak{D}_\kappa(k)$ is a partition in $B_\kappa(k)$.

(7) $\forall k \in \mathbf{n}$: ϕ_k is a function from $\bigcup_{\kappa=1}^{\nu} \mathfrak{D}_\kappa(k)$ into $\bigcup_{\kappa=1}^{\nu} \mathscr{C}_\kappa(k)$, such that $\phi_k(D_\kappa) \in \{C_\kappa \in \mathscr{C}_\kappa(k) \mid C_\kappa \subseteq D_\kappa\}$.

(8) ψ is a function from $\{A_\kappa \in \mathscr{A}_\kappa \mid A_\kappa \subseteq B_\kappa(0)$ for some $\kappa = 1, \ldots, \nu\}$ into $\bigcup_{\kappa=1}^{\nu} \mathscr{C}_\kappa(0)$, such that $\psi(A_\kappa) \in \{C_\kappa \in \mathscr{C}_\kappa(0) \mid C_\kappa \subseteq A_\kappa\}$.

(9) $\forall k \in \{0\} \cup \mathbf{n}$, $\forall \kappa \in \boldsymbol{\nu}$, and for every A_k of $\mathscr{A}_k$ which is a subset of $B_\kappa(k)$, $\tau(A_\kappa)$ is a topology over $\{C_\kappa \in \mathscr{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa\}$ such that $\langle\{C_\kappa \in \mathscr{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa\}, \tau(A_\kappa)\rangle$ is a topological space.

(10) $\forall k \in \{0\} \cup \mathbf{n}, \forall \kappa \in \boldsymbol{\nu}$, and for every A_k of $\mathscr{A}_k$ which is a subset of $B_\kappa(k)$: $\mathfrak{F}(A_k)$ is the minimum σ-algebra generated by the open sets of the topological space $\langle\{C_\kappa \in \mathscr{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa\}, \tau(A_\kappa)\rangle$.

(11) $\forall k \in \{0\} \cup \mathbf{n}, \forall \kappa \in \boldsymbol{\nu}$, and for every A_k of $\mathscr{A}_k$ which is a subset of $B_\kappa(k)$: $P(A_\kappa)$ is a measure over the measurable space $\langle\{C_\kappa \in \mathscr{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa\}, \mathfrak{F}(A_\kappa)\rangle$.

(12) $\forall k \in \mathbf{n}$, d_k is a function from the set $\prod_{k=1}^{n} \prod_{D_\kappa \in \bigcup_\kappa \mathfrak{D}_\kappa(k)} \{C_\kappa \in \mathscr{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa \subseteq D_\kappa\}$ into the set $\prod_{D_\kappa \in \bigcup_\kappa \mathfrak{D}_\kappa(k)} \{C_\kappa \in \mathscr{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa \subseteq D_\kappa\}$.

Conditions defining the partial potential models are only very general basic conditions that the intended applications must satisfy. There are additional *necessary* conditions that a partial potential model must satisfy in order to be a possible application of the theory. In order to motivate these necessary conditions, as well as those defining the models of the theory, I shall discuss in what follows the intended interpretations of the terms just introduced in the previous definition.

The number ν is intended to represent the *length* of the game, that is to say, the number of moves allowed by the rules of the game. This number is held to be constant by means of the convention of taking an upper bound for the number of moves that may possibly take place in any actual play of the game, and then completing any such play by means of dummy moves (if required), so that every play turns out to have exactly the same number of moves.[6]

Ω is the set of all plays of the game. That is to say, in Ω I find all the possible sequences of moves, even those that will turn out to be forbidden by the rules of the game.

$\mathscr{A}_\kappa$ is the *umpire's pattern of information.* Immediately preceding the move $\mathscr{M}_\kappa$, the possible choices are represented by the elements A_κ of $\mathscr{A}_\kappa$. The *umpire's actual information* at (i.e. immediately preceding) the move $\mathscr{M}_\kappa$ is an element A_κ of $\mathscr{A}_\kappa$ whose members are sequences of the form $\sigma_1^*, \ldots, \sigma_{\kappa-1}^*, \sigma_\kappa, \sigma_{\kappa+1}, \ldots \sigma_\nu, \sigma_{\nu+1}$, where the positions $\sigma_1^*, \ldots, \sigma_{\kappa-1}^*$ are constant and σ_κ is a variable that runs throughout all the possible choices that can be taken at $\mathscr{M}_\kappa$ given the previous course of the play $\sigma_1^*, \ldots, \sigma_{\kappa-1}^*$. This implies that the umpire "sees" at every move all the previous positions of the play; in

[6] Cf. VNM, pp. 59-60.

particular, at the end of the game (at $\nu + 1$) the pattern of information of the umpire determines the game fully.

At very A_κ, it is the turn of precisely one player k, not necessarily the same in all A_κs. This is expressed by saying that k is constant in A_κ but different ks are constant in different A_κs. If I take all those A_κs in which k is constant, and form their union, I get $B_\kappa(k)$, which contains all the plays in which it is the turn of player k at position κ. The sequence of all these $B_\kappa(k)$s ($k = 0, 1, \ldots, n$) is denoted as $\mathcal{B}_\kappa$ and called the *pattern of assignment*. The $B_\kappa(k)$ corresponding to the player k whose turn it is, is the *actual assignment* at move $\mathcal{M}_\kappa$.

Suppose, now, that the actual assignment of the move $\mathcal{M}_\kappa$ is a chance move. This means that the choice is within $B_\kappa(0)$. In each of the (pairwise disjoint) A_κs making up $B_\kappa(0)$, both the possible alternatives and the probabilities associated to this alternatives are constant, but they may vary from one set to the other. Within each of those A_κ, the choice among the alternatives takes place (at random); i.e. the choice of a σ_κ. To each σ_κ in a sequence of the form $\sigma_1^*, \ldots, \sigma_{\kappa-1}^*, \sigma_\kappa, \sigma_{\kappa+1}, \ldots, \sigma_{\nu+1}$ there corresponds a set C_κ which contains all sequences of the given form. Thus, the C_κs induce a partition of A_κ for every $A_\kappa \subseteq B_k(0)$.

If the actual assignment of the move $\mathcal{M}_\kappa$ is personal, and so it takes place within $B_\kappa(k)$ ($k \neq 0$), then the state of information of player k must be taken into account. This state of information is represented by a subpartition $\mathcal{D}_\kappa(k)$ of $B_\kappa(k)$ into disjoint sets D_κ. At least the possible alternatives and the information that it is his turn are constant in each of these D_κ, but this information may vary from one D_κ to another. In particular, the possible alternatives must be constant within the D_κ such that $D_\kappa \cap A_\kappa$ is nonempty, where A_κ is the umpire's actual information at $\mathcal{M}_\kappa$. If the choice of k takes place within D_κ, it operates a subdivision $\mathcal{C}_\kappa(k)$ of D_κ into disjoint sets C_κ corresponding to the different alternatives within D_κ. This is the pattern of choice when the move is personal.

The decision functions or strategies ϕ_k and ψ assign a choice to each player at each of his information sets. In particular, ψ assigns to each $A_\kappa \subseteq B_\kappa(k)$ a point (a set) $\psi(A_\kappa)$ in $\{C_\kappa \in \mathcal{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa\}$, where the probability of the point being in set $Y \in \mathfrak{F}(A_\kappa)$ is $P(A_\kappa)(Y)$. For $k \in \mathbf{n}$, ϕ_k assigns to each $D_\kappa \in \mathcal{D}_\kappa(k))$ a subset C_κ of the A_κ which turns out to be the umpire's actual information at $\mathcal{M}_\kappa$ (the player does not have to know which A_κ is this).

At any rate, the decision functions determine a unique play π as the actual course of the game, in a way that I shall describe in what follows.

At $\mathcal{M}_1$ there is only one set A_1, namely Ω, in which a certain k is constant. In this case $\phi_k(A_1)$, or $\psi(A_1)$ for $k = 0$, is equal to a $C_1 \subseteq A_1$. Thus, the umpire's actual information at $\mathcal{M}_2$ is $A_2 = A_1 \cap C_1 \in \mathcal{A}_2$.

Suppose now that at $\mathcal{M}_\kappa$ there is an actual assignment $B_\kappa(k)$; i.e. it is the turn of player $k \neq 0$ at $\mathcal{M}_\kappa$. Actually, $B_\kappa(k) = \bigcup\{A_\kappa \in \mathcal{A}_\kappa \mid k \text{ is constant in } A_\kappa\}$, and $\mathcal{A}_\kappa$ is a subpartition of $\mathcal{D}_\kappa(k)$ within $B_\kappa(k)$. This means that every $A_\kappa \in \mathcal{A}_\kappa$ which is a subset of $B_\kappa(k)$ is also a subset of some D_κ which is also a subset of $B_\kappa(k)$. Now, since $\mathcal{D}_\kappa(k)$ is a partition in $B_\kappa(k)$, *every* $D_\kappa \in \mathcal{D}_\kappa(k)$ is a subset of $B_\kappa(k)$. Hence, every A_κ in which k is constant is included in some information set $D_\kappa \in \mathcal{D}_\kappa(k)$. (Thus, actually, $\bigcup\mathcal{D}_\kappa(k) = B_\kappa(k)$.) For any $D_\kappa \in \mathcal{D}_\kappa(k)$ there is a $C_\kappa \in \mathcal{C}_\kappa(k)$ such that $C_\kappa \subseteq D_\kappa$ (by G10 below). It is also easy to see that (for $A_\kappa \subseteq B_\kappa(k)$), $A_\kappa \cap D_\kappa \neq \varnothing$ implies $A_\kappa \subseteq D_\kappa$ and so that there is an A_κ with

$A_\kappa \subseteq D_\kappa$. By G9 (below), $A_\kappa \cap C_\kappa$ is nonempty. Hence, $\phi(D_\kappa) = C_\kappa$ is well defined and, as a matter of fact, $A_{\kappa+1} = \phi_k(D_\kappa) \cap A_\kappa$ is the umpire's actual information at $\mathcal{A}_{\kappa+1}$. An analogous argument holds if $k = 0$, with $\{A_\kappa \in \mathcal{A}_\kappa \mid A_\kappa \subsetneq B_\kappa(0)$ for some $\kappa = 1, \ldots, \nu\}$ instead of $\mathcal{D}_\kappa(k)$. Clearly, at $\mathcal{A}_{\nu+1}$ a unique course of play π has been determined, which depends upon the decisions or strategies adopted by the players:

$$\pi = \pi(\psi, \phi_1, \ldots, \phi_n).$$

From now on, I shall denote by ϕ_{-k} the $n-1$ tuple $(\phi_1, \ldots, \phi_{k-1}, \phi_{k+1}, \ldots, \phi_n)$, and by Φ_{-k} the set of all these tuples. Also, (ϕ_{-k}, ϕ_k) will denote $(\phi_1, \ldots, \phi_{k-1}, \phi_k, \phi_{k+1}, \ldots, \phi_n)$. If agents other than k and 0 adopt strategies ϕ_{-k}, then they narrow down the set of possible plays given these strategies to the set $\pi(\psi, \phi_{-k}) \equiv \{\phi_k \in \Phi_k \mid \pi(\psi, \phi_{-k}, \phi_k) \in \Omega\}$, in which case k is constrained to adopt a strategy in the set $\varphi_k(\phi_{-k}, \phi_k) = \{\phi_k \in \Phi_k \mid \pi(\psi, \phi_{-k}, \phi_k) \in \pi(\psi, \phi_{-k})$ for some $\psi \in \Psi\}$. Thus, a correspondence $\varphi_k \colon \Phi \to \Phi_k$ is defined for every $k \in \mathbf{n}$.

At a $D_\kappa \in \mathcal{D}_\kappa(k)$, player k has to make a choice among the different alternatives that present before him. The set of these alternatives is actually the κth projection of the A_κ that turns out to be the umpire's actual information at $\mathcal{M}_\kappa$; i.e. the set $\{\sigma_\kappa \mid \sigma_1^*, \ldots, \sigma_{\kappa-1}^*, \sigma_\kappa, \sigma_{\kappa+1}, \ldots, \sigma_{\nu+1} \in A_\kappa\}$. Notice that, since the alternatives are constant at D_κ, the κth projection of any $A_\kappa \subseteq D_k$ is identical to this projection set. To each $A_\kappa \subseteq D_\kappa$ and σ_κ in the projection set there corresponds a $C_\kappa \subsetneq A_\kappa$ such that the κth projection of C_κ is precisely σ_κ. Hence, there is a bijection between the projection set and $\{C_\kappa \mid C_\kappa \subseteq A_\kappa\}$ for any given $A_\kappa \subseteq D_\kappa$. Thus, any given topology on the projection set can be induced over any of these latter sets, so that all of them turn out to be homeomorphic. Therefore, I shall suppose that the topologies $\tau(A_\kappa)$ are "identical" for the A_κs included in the same D_κ, and so there will be no loss of generality when I say that, in adopting an alternative, the player has actually adopted the C_κ corresponding to the alternative which is a subset of the umpire's actual information; i.e. a point in the topological space associated to the umpire's actual information.

From now on, in order to simplify notation a little, I shall adopt the following conventions:

Let $\Xi_k \equiv \bigcup_{\kappa=1}^{\nu} \mathcal{D}_\kappa(k)$.

Let $\Xi \equiv \bigcup_{k=1}^{n} \Xi_k$.

Let $Z \equiv \bigcup_{\kappa=1}^{\nu} \{A_\kappa \in \mathcal{A}_\kappa \mid A_\kappa \subseteq B_\kappa(0)$ for some $\kappa = 1, \ldots, \nu\}$.

$S_\xi \equiv \{C_\kappa \in \mathcal{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa \subseteq D_\kappa\}$, where A_κ is the umpire's actual information at $\mathcal{M}_\kappa$ for some κ and $\xi = D_\kappa \in \mathcal{D}_\kappa(k)$.

$S_\zeta \equiv \{C_\kappa \in \mathcal{C}_\kappa(0) \mid C_\kappa \subseteq A_\kappa\}$, where $\zeta = A_\kappa$.

$\tau_\xi \equiv \tau(A_\kappa)$, where $\xi = D_\kappa$, $A_\kappa \subseteq D_\kappa$, and A_κ is the umpire's actual information at $\mathcal{M}_\kappa$.

$\tau_\zeta \equiv \tau(A_\kappa)$, $\zeta = A_\kappa$.

$\mathfrak{F}_\xi$ is the minimum σ-algebra generated by the open sets in τ_ξ.

$\mathfrak{F}_\zeta \equiv \mathfrak{F}(A_\kappa)$, where $\zeta = A_\kappa$.

$\Phi_k \equiv \prod_{\xi\in\Xi_k} S_\xi$, namely the set

$$\left\{\phi_k : \Xi_k \to \bigcup_{\xi\in\Xi_k} S_\xi \mid \phi_k(\xi) \in S_\xi\right\}.$$

$\Psi \equiv \prod_{\zeta\in Z} S_\zeta$, namely the set

$$\left\{\psi : Z \to \bigcup_{\zeta\in Z} S_\zeta \mid \psi(\zeta) \in S_\zeta\right\}.$$

$\Phi \equiv \prod_{k=0}^{n} \Phi_k$.

Function d_k describes the decision actually adopted by player k; i.e. $d_k(\phi_{-k}, \phi_k)$ is the point chosen by agent k in his space ϕ_k of possible strategies, given that personal players other than k chose strategies ϕ_{-k}.

3. THE DOMAIN OF INTENDED APPLICATIONS

Given the tremendous variety of games in the literature, that I mentioned in the introduction, it would be futile to attempt to provided here a list of the intended applications of the theory. On the other hand, this is a topic that is hardly treated in the literature. Usually, a new type of game is presented without a really careful attempt to discuss its empirical import. The aim of this section is to provide a set of *necessary* conditions that the intended applications must satisfy in order to be considered as such. These conditions define a subset **R** of $\mathbf{M}_{pp}$(**GAME**) Roughly, the behavior these conditions describe is one of abiding to rules. Whether this rule-obeying behavior turns out to be a utility-maximization behavior is quite another matter: this is the empirical claim of the theory.

More precisely, the empirical claim of game theory is not that *any* rule-obeying behavior is utility-maximization behavior, but only that *certain* cases of rule-obeying behavior (i.e. some elements of **R**) are so. How to characterize these elements (a task that involves a lot of pragmatic and historic elements) is a question that cannot be tackled here. The level of generality at this point only requires that I accept the existence of some set $\mathbf{I} \subseteq \mathbf{R}$ containing these elements. This set will be called the *domain of intended applications*. The set **R** is defined as the extension of the following set-theoretic predicate.

DEFINITION 2: $x = \langle \nu, n, \Omega, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \phi, \psi, \tau, \mathfrak{F}, P, d\rangle$ is a *ruled-behavior structure* iff $x \in \mathbf{M}_{pp}$(**GAME**) and x satisfies the following axioms:

(G1) $\mathcal{A}_\kappa$ is a subpartition of $\mathcal{B}_\kappa$.

(G2) $\mathcal{C}_\kappa(0)$ is a subpartition of $\mathcal{A}_\kappa$.

(G3) $\forall k \in \mathbf{n}$: $\mathcal{C}_\kappa(k)$ is a subpartition of $\mathcal{D}_\kappa(k)$.

(G4) $\forall k \in \mathbf{n}$: Within $B_\kappa(k)$, $\mathcal{A}_\kappa$ is a subpartition of $\mathcal{D}_\kappa(k)$.

(G5) $\forall k \in \{0\} \cup \mathbf{n}, \forall \kappa \in \boldsymbol{\nu}$, and for every A_k of $\mathcal{A}_k$ which is a subset of $B_\kappa(k)$: $P(A_\kappa)$ is a probability measure over the measurable space $\langle \{C_\kappa \in \mathcal{C}_\kappa(k) \mid C_\kappa \subseteq A_\kappa\}, \mathfrak{F}(A_\kappa) \rangle$.

(G6) $\mathcal{A}_1$ consists of the one set Ω.

(G7) $\mathcal{A}_{\nu+1}$ consists of singletons.

(G8) $\forall \kappa \in \boldsymbol{\nu}$: $\mathcal{A}_{\kappa+1}$ obtains from $\mathcal{A}_\kappa$ by superposing it with all $\mathcal{C}_\kappa(k)$ ($k = 0, 1, \ldots, n$).

(G9) $\forall \kappa \in \boldsymbol{\nu}$: If A_κ of $\mathcal{A}_\kappa$ and C_κ of $\mathcal{C}_\kappa(k)$ ($k = 1, \ldots, n$) are subsets of the same D_κ of $\mathcal{D}_\kappa(k)$, then the intersection $A_\kappa \cap C_\kappa$ must not be empty.

(G10) $\forall \kappa \in \boldsymbol{\nu}, \forall k \in \mathbf{n}$, and every D_κ of $\mathcal{D}_\kappa(k)$: Some C_κ of $\mathcal{C}_\kappa(k)$, which is a subset of D_κ, must exist.

The intended interpretation of axioms (G1)-(G10) is the following.

(G1*) The umpire's pattern of information at the move $\mathcal{M}_\kappa$ includes the assignment of that move.

(G2*) The pattern of choice at a chance move $\mathcal{M}_\kappa$ includes the umpire's pattern of information at that move.

(G3*) The pattern of choice at a personal move $\mathcal{M}_\kappa$ of the player k includes the player k's pattern of information at that move.

(G4*) The umpire's pattern of information at the move $\mathcal{M}_\kappa$ includes –to the extent to which this is a personal move of the player k– the player's k pattern of information at that move.

(G5*) The various alternative choices at a chance move $\mathcal{M}_\kappa$ constitute the elementary events of a certain probability space.

(G6*) The umpire's pattern of information at the first move is void.

(G7*) The umpire's pattern of information at the end of the game determines the play fully.

(G8*) The umpire's pattern of information at the move $\mathcal{M}_{\kappa+1}$ (for $\kappa = \nu$: at the end of the game) obtains from that one at the move $\mathcal{M}_\kappa$ by superposing it with the pattern of choice at the move $\mathcal{M}_\kappa$.

(G9*) Let a move $\mathcal{M}_\kappa$ be given, which is a personal move of the player k, and any actual information of the player k at that move also be given. Then any actual information of the umpire at that move and any actual choice of the player k at that move, which are both within (i.e. refinements of) this actual (player's) information, are also compatible with each other. I.e. they occur in actual plays.

(G10*) Let a move $\mathcal{M}_\kappa$ be given, which is a personal move of the player k, and any actual information of the player k at that move also be given. Then the number of alternative actual choices, available to the player k, is not zero.

4. THE MODELS OF THE THEORY

In a first approach, the empirical claim of **GAME** is that the behavior described by the elements of **I** is utility-maximizing behavior. In order to make precise this claim, I need to formulate the fundamental law of game theory. This, in turn, requires the introduction of the following conceptual apparatus

In order to define the objective function, which is not exactly bare utility, but rather expected utility, I need to define before the umpire's experiment. When the index set Z is finite, the umpire's experiment is just the direct product of the probability spaces $(S_\zeta, \mathfrak{F}_\zeta, P_\zeta)$. In the general case, the umpire's experiment is the projective limit of the family of all such possible direct products, as characterized in what follows.

Let D be the class of all finite subsets of Z directed by inclusion; i.e. $\alpha < \beta$ iff $\alpha \subseteq \beta$ for $\alpha, \beta \in D$. For each $\alpha \in D$ and family $\{(S_\zeta, \mathfrak{F}_\zeta, P_\zeta)\}_{\zeta \in \alpha}$ of probability spaces, I let $(S_\alpha, \mathfrak{F}_\alpha, P_\alpha)$ be the direct product of these spaces. For every $\alpha < \beta$ $(\alpha, \beta \in D)$, let $g_{\alpha\beta}: S_\beta \to S_\alpha$ be the coordinate projection, namely the function such that $g_{\alpha\beta}(s_1, \ldots, s_\alpha, \ldots, s_\beta) = (s_1, \ldots, s_\alpha)$. It is easy to see that (i) $g_{\alpha\beta}$ is measurable, in the sense that $g_\alpha^{-1}(Y) \in \mathfrak{F}_\beta$ for all $Y \in \mathfrak{F}_\alpha$; (ii) the $g_{\alpha\beta}$s are compatible in the sense that, for $\alpha < \beta < \gamma$, $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$ and $g_{\alpha\alpha}$ is the identity; and $P_\alpha = P_\beta \circ g_{\alpha\beta}^{-1}$ for each $\alpha < \beta$. Hence, the structure $\{(S_\alpha, \mathfrak{F}_\alpha, P_\alpha, g_{\alpha\beta})_{\alpha<\beta} \mid \alpha, \beta \in D\}$ is called a ***projective system of probability spaces***.

The probability space $(S, \mathfrak{F}, P)$ is called the ***projective limit*** of the projective system $\{(S_\alpha, \mathfrak{F}_\alpha, P_\alpha, g_{\alpha\beta})_{\alpha<\beta} \mid \alpha, \beta \in D\}$ if (i) $S = \prod_{\zeta \in Z} S_\zeta$; (ii) $\mathfrak{F}$ is the minimun σ-algebra generated by all cylinder sets of the form $g_\alpha^{-1}(Y_\alpha)$ $(g_\alpha(s) = (s_{\zeta_1}, \ldots, s_{\zeta_m})$ for every $\alpha = \{\zeta_1, \ldots, \zeta_m\} \in D)$, where Y_α is a Borel set in $\mathfrak{F}_\alpha$; (iii) P is the (unique) σ-additive extension to $\mathfrak{F}$ of the measure P_0 defined on the cylinder sets by $P_0(g_\alpha^{-1}(Y_\alpha)) \equiv P_\alpha(Y_\alpha)$.

The umpire's experiment is not the only projective limit required by **GAME**. In fact, also the definition of mixed strategy requires this concept. By the ***umpire's experiment*** I mean precisely the projective limit of the projective system $\{(S_\alpha, \mathfrak{F}_\alpha, P_\alpha, g_{\alpha\beta})_{\alpha<\beta} \mid \alpha, \beta \in D\}$, where D is the family of all finite subsets of Z. By a ***distributional strategy of personal agent k*** I mean, analogously, the projective limit of the system $\{(S_\alpha, \mathfrak{F}_\alpha, P_\alpha, g_{\alpha\beta})_{\alpha<\beta} \mid \alpha, \beta \in D\}$, where D is the family of all subsets of Ξ_k.[7]

From now on, the umpire's experiment shall be denoted as $(\Psi, \mathfrak{G}, Q)$, whereas the distributional strategy of agent k shall be written as $(\Phi_k, \mathfrak{F}_k, P_k)$. Notice that in the case when the space $(\Phi_k, \mathfrak{F}_k, P_k)$ is atomic (i.e. when the singletons $\{\phi_k\} \in \mathfrak{F}_k$ have

[7] Notice that our distributional strategy is similar to what Milgrom and Weber (1985) call a distributional strategy, the difference being that our concept is more general in two respects: (i) the measure P (their μ_i) is not defined over the whole set $\Xi_k \times \bigcup_{\xi \in \Xi_k} S_\xi$ (their $T_i \times A_i$) but only over the set of all functions from Ξ_i into $\bigcup_{\xi \in \Xi_k} S_\xi$; i.e. over Φ_k, which is more natural. (ii) I am not assuming that the Ξ_ks are complete, separable metric spaces.

positive probabilities and $\sum_{\phi_k \in \Phi_k} P(\{\phi_k\}) = 1$), P_k determines a mixed strategy, defined by the distribution $p(\phi_k) = P_k(\{\phi_k\})$. At any rate, as Milgrom and Weber (1985) have shown, "distributional strategies are simply another way of representing mixed and/or behavioral strategies".[8]

Thus, the question whether the distributional strategies and the umpire's experiment are well defined boils down to the question whether the projective limits of the spaces $\{(S_\alpha, \mathfrak{F}_\alpha, P_\alpha, g_{\alpha\beta})_{\alpha<\beta} \mid \alpha, \beta \in D\}$ do exist.

The temptation arises here to find conditions guaranteeing the existence of distributional strategies and the umpire's experiment. Yet, such a search is wrong when the task is to provide the most general formulation of **GAME**. I suspect that any attempt at finding conditions for guaranteeing the existence of such spaces will land in a set of at most *sufficient* conditions to the effect. Actually, the most general result known about the existence of projective limits of probability spaces is Kolmogorov's-Bochner Theorem. According to this theorem, the existence of the required projective limits is implied by two conditions: (i) the topological spaces generating the σ-algebras are Hausdorff, and (ii) the probability measures of the coordinate spaces are Radon measures.[9] Yet, the condition that the measures be Radon is sufficient but not necessary, and so it is actually *restrictive*. For instance, the real spaces dealt with by Kolmogorov in his less general version of the Fundamental Theorem need not be Radon! Therefore, it is more reasonable to assume that the required projective limits are **GAME**-theoretic.

The previous discussion motivates the introduction of the following definition.

DEFINITION 3: $x = \langle \nu, n, \Omega, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \phi, \psi, \tau, \mathfrak{F}, P, d, P^*, \mathfrak{u}\rangle \in \mathbf{M}_p(\mathbf{GAME})$ iff

(1) $\langle \nu, n, \Omega, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \phi, \psi, \tau, \mathfrak{F}, P, d\rangle \in \mathbf{M}_{pp}(\mathbf{GAME})$.

(2) $P_0^* \equiv Q$ is a probability measure over the measurable space $(\Psi, \mathfrak{G})$ such that $(\Psi, \mathfrak{G}, Q)$ is the umpire's experiment and, for every $k \in \mathbf{n}$, $P_k^* \equiv P_k$ is a probability measure over the measurable space $(\Phi_k, \mathfrak{F}_k)$ such that $(\Phi_k, \mathfrak{F}_k, P_k)$ is a distributional strategy of agent k.

(3) For every $k \in \mathbf{n}$, $\mathfrak{u}_k: \Psi \times \Phi \to \mathbb{R}$ is a bounded function such that the restriction of $\mathfrak{u}_k$ to $\Psi \times \{\phi\}$ is $\mathfrak{G}_k$-measurable.

The previous definition – the definition of the potential models of **GAME** – has placed us in good shape to discuss the fundamental law of the theory. This requires the introduction of the crucial notion of expected utility. Notice that since every $(\psi, \phi) \in \Psi \times \Phi$ determines a unique play $\pi = \pi(\psi, \phi) \in \Omega$, and every $\pi \in \Omega$ is determined by a unique strategy $(\psi, \phi) \in \Psi \times \Phi$, I shall write "$\mathfrak{u}_k(\psi, \phi)$" instead of "$\mathfrak{u}_k(\pi(\psi, \phi))$."

DEFINITION 4: The *expected utility function* of personal player $k \in \mathbf{n}$ is the mapping $U_k: \Phi \to \mathbb{R}$ defined for any $\phi \in \Phi$, by the condition

$$U_k(\phi) \equiv \int_{\psi \in \Psi} \mathfrak{u}_k(\psi, \phi)\, dQ.$$

[8] In the sense of Aumann (1964). See p. 620.

[9] A measure over a measurable space is called a *Radon measure* iff the measure of any measurable set is the limit of a sequence of measures of compact measurable sets. See Kolmogorov (1956) and Bochner (1955). A proof of this theorem can be found, for instance, in Rao (1981), pp. 9-12.

For any personal player k, I define the maximization correspondence $\mu_k: \Phi \to \Phi_k$ by means of condition

$$\mu_k(\phi) \equiv \left\{ \phi'_k \in \varphi_k(\phi) \mid U_k(\phi_{-k}, \phi'_k) = \max_{\phi''_k \in \varphi_k(\phi_{-k}, \phi'_k)} U_k(\phi_{-k}, \phi''_k) \right\}.$$

The global maximization correspondence is just $\mu: \Phi \to \Phi$, given by

$$\mu(\phi) \equiv \mu_1(\phi) \times \cdots \times \mu_n(\phi).$$

Analogously, the global decision correspondence is given by

$$d(\phi) \equiv d_1(\phi) \times \cdots \times d_n(\phi).$$

Using this terminology, the models of **GAME** can be defined as follows.

DEFINITION 5: $x = \langle v, n, \Omega, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \phi, \psi, \tau, \mathfrak{F}, P, d, P^*, \mathfrak{u} \rangle \in$ **M(GAME)** iff every personal player maximizes expected utility, i.e. there is a $\phi \in \Phi$ such that

$$d(\phi) \in \mu(\phi).$$

Thus, to say that a certain potential model x is actually a model is tantamount to saying that x has a Cournot-Nash equilibrium. This is the real import of the existence of equilibria. Only potential games that have Cournot-Nash equilibria are actually games.

REFERENCES

Aumann, R.J. (1961). Borel Structures for Function Spaces. In: *Illinois Journal of Mathematics*, **5**, 614-630.

______. (1964). Mixed and Behavior Strategies in Infinite Extensive Games. In: M. Dresher, L.S. Shapley and A.W. Tucker. *Advances in Game Theory*. (*Annals of Mathematics Studies*, **52**). Princeton: Princeton University Press.

Balzer, W., C.U. Moulines, J.D. Sneed. *An Architectonic for Science. The Structuralist Program*. Dordrecht: Reidel, 1987.

Bonnano, G. (1991). Extensive Forms and Set-Theoretic Forms. In: *Economics Letters*, **37**, 363-370.

______. (1993). The Logical Representation of Extensive Games. In: *International Journal of Game Theory*, **22**, 153-169.

Kolmogorov, A.N. (1956). *Foundations of the Theory of Probability*. New York: Chelsea.

Kuhn, H.W. (1953). Extensive Games and the Problem of Information. In: H.W. Kuhn and A.W. Tucker (Eds.), *Contributions to the Theory of Games II*. (*Annals of Mathematics Studies*, **28**). Princeton: Princeton University Press.

McKinsey, J.C.C. (1952). *Introduction to the Theory of Games*. New York: McGraw Hill.

Milgrom, P.R. and R.J. Weber. (1985). Distributional Strategies for Games with Incomplete Information. In: *Mathematics of Operations Research*, **10**, 619-632.

Rao, M.M. (1981). *Foundations of Stochastic Analysis*. New York: Academic Press.

von Neumann, J. and O. Morgenstern. (1944). *Theory of Games and Economic Behavior*. Princeton: Princeton University Press.