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David Mayer Foulkes

HOMOGENIZED INTEGRAL U-STATISTICS FOR TEST OF NON-LINEARITY

Introduction

The interest in statistics capable of detecting non-linear dynamics is now well established in economics. Developing Grassberger and Procaccia's (G&P) (1983) Correlation Dimension (CD), Brock et al. (BDS) defined a statistic testing the IID null whose applications include testing for non-linearity in stochastic processes. Combining these two approaches Mayer (1995, 1996) defined the Correlation Dimension Ratio (CDR) (or Statistical Correlation Dimension), a statistic which tests the IID null, calculates dimensions greater than 1, and eliminates a downward bias present in the G&P and BDS statistics. In a parallel development, Mizrach (1991) defined the Simple Nonparametric Test (SNT), a simpler version of these U-statistics which can be applied for the same purposes and involves less calculation.

The numerical methods introduced by Mayer (1995) to calculate the building block distance histogram $C(m, \varepsilon)$ used in these statistics obtains it for many distance values ε simultaneously, and recursively in the dimension m (see the definitions in the next section), leading to the question wether the information thus obtained can be used more effectively. The first purpose of this paper is to define some *integral U-statistics*, which take averages along the ε variable. In addition, a homogenization process is introduced after which these statistics have distributions independent of the stationary process being tested. The objective is to define statistics for which confidence intervals can be obtained universally, either by theoretical means or by Monte-Carlo experiments.

Although U-statistics can be proved to be asymptotically normal, it cannot be assumed that this convergence is fast enough for empirical purposes, especially in applications in which data availability is relatively low, a situation which typically holds for Economic applications. Also the calculation of the asymptotic variance is itself very lengthy—once the pretty complicated algebraic formulae are obtained. Thus in practice we are interested in the power of tests using confidence intervals obtained by boot-strapping methods which originate in Brock's reshuffling test. The second purpose of this paper is to define some particular integral statistics and evaluate the corresponding reshuffling tests using a Monte-Carlo experiment. For purposes of comparability we use for this experiment the non-linear series which Barnett et. al. (1996) used in their double blind experiment on test of non-linearity.

The sections of this paper are organized as follows. In the first, we review the definitions of the SNT, BDS, CD, CDR and some related statistics, and introduce notation. In the second, we define the process of homogenization alluded to before. In the third, we define *homogenized integral U-statistics*. In the fourth, we define some particular integrals. In the fifth, we describe the Monte-Carlo experiments and report its results. Then we offer some concluding remarks.

SNT, BDS, and related statistics

Let $\mathbf{Z}^p = (Z_1^p, ..., Z_m^p)$, $p = 1, ..., N$ be N copies of an m-dimensional multivariate random variable $\mathbf{Z}$. Let I be the indicator function,

$$I(x,y) = \begin{cases} 1 & x \leq y, \\ 0 & x > y. \end{cases} \tag{1}$$

Define the *Order 1* "building block" random variables,

$$c_i^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = \frac{1}{N} \sum_{p=1}^{N} I(\,|Z_i^p - z_{0i}|\,,\ \varepsilon),\ i = 1, ..., m, \tag{2}$$

$$C^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = \frac{1}{N} \sum_{p=1}^{N} I(\max_{1 \leq i \leq m} |Z_i^p - z_{0i}|\,,\ \varepsilon). \tag{3}$$

These random variables can be used to define a whole family of statistics. One example is the Simple Non-parametric Test (SNT statistic, Mizrach, 1991)

$$SNT(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = C^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)\ -\ \prod_{1 \leq i \leq m} c_i^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N). \tag{4}$$

Another is the Order 1 Ratio Statistic (RS^1 statistic)

$$RS^1\ (\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = C^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)\ /\ \prod_{1 \leq i \leq m} c_i^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N). \tag{5}$$

We can also define two local dimension measures: the Correlation Dimension (CD^1), and the Correlation Dimension Ratio (CDR^1) at $\mathbf{z}_0$. We first define

$$CD^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = \ln(C^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)\,)\ /\ \ln(\varepsilon), \tag{6}$$

$$CDR^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)\ =\ \ln(C^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)\,)\ /\ \sum_{1 \leq i \leq m} \ln(c_i^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)), \tag{7}$$

and then write

$$CD^1(\mathbf{Z}, \mathbf{z}_0) = \lim_{\varepsilon \to 0} \lim_{N \to \infty}\ CD^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)\ = \lim_{\varepsilon \to 0}\ E(CD^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)), \tag{8}$$

and similarly for CDR^1.

Now define the *Order 2* "building block" random variables

$$c_i^2(\mathbf{Z}, \varepsilon, N) = \frac{2}{N(N-1)} \sum_{p<q} I(|Z_i^p - Z_i^q|\,,\ \varepsilon),\ i = 1, ..., m, \tag{9}$$

$$C^2(\mathbf{Z}, \varepsilon, N) = \frac{2}{N(N-1)} \sum_{p<q} I(\max_{1 \leq i \leq m} |Z_i^p - Z_i^q|\,,\ \varepsilon). \tag{10}$$

The BDS (Brock, Dechert, Scheinkman) statistic is

$$BDS\ (\mathbf{Z}, \varepsilon, N)\ = C^2(\mathbf{Z}, \varepsilon, N)\ -\ \prod_{1 \leq i \leq m} c_i^2(\mathbf{Z},\ \varepsilon,\ N). \tag{11}$$

This can be modified, for example, to the Order 2 Ratio Statistic (RS^2 statistic)

$$RS^2\ (\mathbf{Z},\varepsilon, N)\ = C^2(\mathbf{Z},\varepsilon, N)\ /\ \prod_{1\leq i\leq m} c_i^2(\mathbf{Z},\ \varepsilon,\ N). \tag{12}$$

The (non-local) Correlation Dimension (CD) defined by Grassberger Procaccia is given by the limit

$$CD^2(\mathbf{Z},\varepsilon, N)\ = \ln\ (C^2(\mathbf{Z},\varepsilon, N))\ /\ \ln(\varepsilon). \tag{13}$$

as $N \to \infty$ and $\varepsilon \to 0$. The (non-local) Correlation Dimension Ratio (CDR) studied by Mayer (1995) and Mayer and Feliz (1996) is instead

$$CDR^2(\mathbf{Z},\varepsilon, N)\ =\ \ln(C^2(\mathbf{Z},\varepsilon, N))/ \sum_{1\leq i\leq m} \ln(c_i^2(\mathbf{Z},\varepsilon, N)). \tag{14}$$

We shall write

$$CD^2(\mathbf{Z}) = \lim_{\varepsilon\to 0}\lim_{N\to\infty} CD^2(\mathbf{Z},\varepsilon, N)\ = \lim_{\varepsilon\to 0}\ E(CD(\mathbf{Z},\varepsilon, N)), \tag{15}$$

and similarly for the CDR. In practice the GP and CDR statistics are usually calculated as regressions of $C^2(\mathbf{Z},\varepsilon, N)$ in terms of $C^2(\mathbf{Z},\varepsilon, N)$ or $\ln(\varepsilon)$ for small values of ε . These are thus more complicated functions of the building block random variables.

Theorem 1 *Let Z_i be a strictly stationary process which is absolutely regular. (A definition is omitted for brevity. There are alternative conditions on the rate of decay of dependence over time yielding the same result, See Denker and Keller, 1983, p. 507). Generically, the building block statistics and smooth functions of them such as the SNT, BDS, RS^j, CD^j and CDR^j statistics are asymptotically normal as $N \to \infty$. The later statistics have means*

$$E(SNT) = 0,\quad E(BDS) = 0,\quad E(RS^j) = 1,\quad E(CDR^j) = 1, \tag{16}$$

($j = 1, 2$) for any $\varepsilon > 0$ and for any $\mathbf{z}_0$ satisfying $E(\mathbf{I}(\,|Z_i - z_{0i}|\,, \varepsilon)) > 0,\ i = 1, ..., m$. The asymptotic variance of the order j statistics depends on the variances and covariances of the building block random variables c_i^j, C^j, $i = 1, ..., m,\ j = 1, 2$. ■

Homogenization of multivariate random variables

We shall write $\mathfrak{U} : [0, 1] \to [0, 1]$ and $\mathfrak{N}\!: \mathbb{R}^E \to [0, 1]$ for the accumulated density functions of the standard uniform and normal distributions respectively,

$$\mathfrak{U}\ (z) =\ z,\ \mathfrak{N}\ (z) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{z} \exp(-t^2)dt, \tag{17}$$

where $\mathbb{R}^E = \mathbb{R} \bigcup \{-\infty, \infty\}$ is the extended real line (with the one-point compactification topology on each extreme). We can thus write $\mathfrak{N}\ (\pm\infty)$, $\mathfrak{N}^{\ -1}(0)$, $\mathfrak{N}^{\ -1}(1)$, and write about the uniform and normal distributions in the same terms.

Suppose that $\mathbf{Z}$ has the accumulated density function

$$P(Z_i \leq z_i,\ i = 1, ..., m) = F(z_1, ..., z_m). \tag{18}$$

Definition 1 We shall say that $\mathbf{Z}$ is *ammenable to homogenizations* if each function

$$p_i(z) = P(Z_i \leq z),\ i = 1, ..., m \tag{19}$$

is continuous and surjective from its domain of definition (possibly $\mathbb{R}^E$) to [0, 1].■

For any continuous, increasing surjective functions $G_i : I_i \to [0, 1]$, where $I_i \subseteq \mathbb{R}^E$are closed intervals, $i = 1, ..., m$ define the random variables

$$X_i = (G_i^{-1} \circ p_i)(Z_i),\ i = 1, ..., m, \tag{20}$$

where G_i^{-1} is the increasing, semi-continuous function satisfying $G \circ G_i^{-1} = id$ defined by $G_i^{-1}(y) = \inf\{x\,|G(x) = y\}$. Observe that the random variables X_i have accumulated density function G_i, since

$$P(X_i \leq x) = P(G_i^{-1}(p_i(Z_i)) \leq x) = P(Z_i \leq p_i^{-1}(G_i(x))) = p_i(p_i^{-1}(G_i(x))) = G_i(x) \tag{21}$$

(where p_i^{-1} are defined as G_i^{-1}). We thus have the following definition.

Definition 2 Write $\mathbf{G} = (G_1, ..., G_m)$. The multivariate random variable just defined, $\mathbf{X} = (X_1, ..., X_m)$ is the $\mathbf{G}$-homogenization of $\mathbf{Z}$, and we write $X = \mathfrak{H}_G(Z)$.■

In particular, if G_i is $\mathfrak{U}$ or $\mathfrak{N}$, X_i is uniformly distributed on $[0, 1]$ or follows the standard normal distribution.

In the following theorem we show that in some special cases the homogenization map $\mathfrak{H}_G$ leaves the order 1 statistics invariant, while in the general case the topological properties of the order 1 and order 2 statistics are preserved. We write $c^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = c_i^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N)$ when Z_i are identical random variables.

Theorem 2 *If $Z_1, ..., Z_m$ are identical random variables, $G_1 = ... = G_m = G$, and $\mathbf{z}_0 = 0$ or $\mathbf{z}_0 = (1, ..., 1)$, then the building block random variables are invariant under homogenization:*

$$c^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = c^1(\mathbf{X}, \mathbf{x}_0, G^{-1}(p(\varepsilon)), N), \tag{22}$$

$$C^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = C^1(\mathbf{X}, \mathbf{x}_0, G^{-1}(p(\varepsilon)), N), \tag{23}$$

where $\mathbf{x}_0 = \varphi(\mathbf{z}_0)$ and p is any of the functions p_i defined above. Thus the order 1 statistics are preserved under the homogenization map $\mathfrak{H}_G$ if ε is transformed accordingly.

Suppose in the general case that the functions p_i defined above and $\mathbf{G}$ are diffeomorphisms. Then the homogenizing transformation $\mathfrak{H}_G$ is equivalent to applying the diffeomorphism $\varphi = (G_1^{-1} \circ p_{1,...,} G_m^{-1} \circ p_m)$. Hence the dimension measures CD^j, CDR^j, $j = 1, 2$, satisfy

$$CD^1(\mathbf{Z}, \mathbf{z}_0) = CD^1(\mathfrak{H}_G(\mathbf{Z}), \varphi(\mathbf{z}_0)),\quad CD^2(\mathbf{Z}) = CD^2(\mathfrak{H}_G(\mathbf{Z})), \tag{24}$$

$$CDR^1(\mathbf{Z}, \mathbf{z}_0) = CDR^1(\mathfrak{H}_G(\mathbf{Z}), \varphi(\mathbf{z}_0)),\quad CDR^2(\mathbf{Z}) = CDR^2(\mathfrak{H}_G(\mathbf{Z})). \tag{25}$$

Proof: The first statement follows from the equalities

$$I(Z_i,\varepsilon)=I(X_i,G^{-1}(p(\varepsilon))),\ I(\max_{1\le i\le m} Z_i,\varepsilon)=I(\max_{1\le i\le m} X_i,\ G^{-1}(p(\varepsilon))). \tag{26}$$

The second statement is an application of Brock and Dechert's results (1988).■

Homogenized integral U-statistics

The purpose of this paper is to explore the statistical properties of variants of the SNT and BDS statistics when they are applied *after* one of the homogenization processes $\mathfrak{H}_{\mathfrak{U}}$ or $\mathfrak{H}_{\mathfrak{N}}$, in which the component random variables are transformed into uniform or normal random variables. Theorem 2.3 shows that these statistics will tend to detect the same properties as the original ones, and thus we consider them in their own right.

The idea is the following. After the random variables $\mathbf{Z}$ are transformed to $\mathbf{X}$, each X_i becomes a uniform or normal random variable. Thus the properties of these homogenized statistics will no longer depend on the particular distributions F_i of Z_i. Indeed, under not very stringent conditions the distributions $C^1(\mathbf{X},\mathbf{x}_0,\varepsilon,N)$ and $C^2(\mathbf{X},\varepsilon,N)$ will be known in the IID case and thus the confidence intervals of the *homogenized* statistics will be simpler to obtain. In the more general case in which these distributions are unknown, results derived by Monte-Carlo methods will be simultaneously applicable to all amenable distributions $\mathbf{Z}$.

The statistics we shall consider will be *integral* because we shall consider different kinds of averages along ε. This we do to use more of the information contained in the building block functions, which is in any case obtained at no extra expense when the calculations are actually performed.

Suppose Z has identical component random variables $Z_1,...,Z_m$. We begin by giving an example which will clarify the motivation behind our definitions. In the study of the BDS statistic the null hypothesis is

$$E(C^2(\mathbf{Z},\varepsilon,N))=E(c^2(\mathbf{Z},\varepsilon,N))^m. \tag{27}$$

As mentioned above, one may carry out a regression for low values of to find d such that

$$\ln(C^2(\mathbf{Z},\varepsilon,N))=const+d\ \ln(c^2(\mathbf{Z},\varepsilon,N)), \tag{28}$$

and then test in some way if d is less than m or not. A natural question is to ask how one can get additional information from $C^2(\mathbf{Z},\varepsilon,N)$ (which can be calculated at no extra expense for the full range of ε simultaneously). Writing

$$C^2(\mathbf{Z},\varepsilon)=\lim_{N\to\infty}C^2(\mathbf{Z},\varepsilon,N)=E(C^2(\mathbf{Z},\varepsilon,N)), \tag{29}$$

and similarly for $c^2(\mathbf{Z},\varepsilon)$, integrals such as the following are attractive:

$$\mathfrak{I}^j(\mathbf{Z},\mathbf{z}_0)=\int_{c^2(\mathbf{Z},\varepsilon)\in[a,b]}\left(\frac{C^j(\mathbf{Z},\mathbf{z}_0,\varepsilon)}{C^j(\mathbf{Z},\mathbf{z}_0,\varepsilon)^m}-1\right)^p dc^j(\mathbf{Z},\mathbf{z}_0,\varepsilon), \tag{30}$$

where we now consider $j=1$ or 2, and it is understood that in the case $j=2$ there is no dependence on $\mathbf{z}_0$. Other variations can be written down, involving logarithms and other functional forms. Here we use an L^p norm for the sake of example. The measure $dc^j(\mathbf{Z}, \mathbf{z}_0, \varepsilon)$ is attractive because it is proportional to the number of events at ε. Notice that, in the IID case, $E(\mathfrak{I}^j(\mathbf{Z}, \mathbf{z}_0) = 0$. In the case of the order1 SNT-type statistics,by using the change of variable $\tau = G^{-1}(p(\varepsilon))$, we obtain $\mathfrak{I}^1(\mathbf{Z}, \mathbf{z}_0) = \mathfrak{I}^1(\mathbf{X}, \mathbf{x}_0)$. In the case of the order 2 BDS-type statistics, this transformation does not go through, but we shall first use the diffeomorphism $\varphi = (G_1^{-1} \circ p_{1,...,} G_m^{-1} \circ p_m)$ to transform $\mathbf{Z}$ to $\mathbf{X}$ and then apply $\mathfrak{I}^2$to $\mathbf{X}$.

We now construct a generalized instance of this type of integral, approximated in the summation form. (Recall our assumption that $Z_{1,...,} Z_m$ are identical random variables.) We begin with the case of order one statistics, first defining the more primitive "building block" random variables:

$$h^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = \frac{1}{N} \sum_{p=1}^{N} \mathbf{I}(Z_i^p - z_{0i}, \varepsilon), \tag{31}$$

$$H^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = \frac{1}{N} \sum_{p=1}^{N} I(\max_{1 \le i \le m} (Z_i^p - z_{0i}), \varepsilon). \tag{32}$$

(where, in comparison to c_i^1 we have omitted the absolute values; we use any $i = 1, ...m$). As in Theorem 2, if $\mathbf{x}_0 = (G^{-1} \circ p)(\mathbf{z}_0)$,

$$h^1(\mathbf{Z}, \mathbf{z}_0, \varepsilon, N) = h^1(\mathbf{X}, \mathbf{x}_0, (G^{-1} \circ p)(\varepsilon), N), \tag{33}$$

and similarly for c^1, H^1. Let $\mathfrak{B}^1 = \{c^1, h^1, C^1, H^1\}$ be the set of building block random variables. Given a partition $\varepsilon_0 < ... < \varepsilon_I$ of the interval $[\varepsilon_0, \varepsilon_I]$ ($I \in \mathbb{N}$), let

$$\Delta g^1(\mathbf{Z}, \mathbf{z_0}, \varepsilon_k, N) = g^1(\mathbf{Z}, \mathbf{z_0}, \varepsilon_k, N) - g^1(\mathbf{Z}, \mathbf{z_0}, \varepsilon_{k-1}, N) \; for \; g^1 \in \mathfrak{B}^1. \tag{34}$$

($k = 1, ..., I$). Write $\mathfrak{b}^1$ for (c^1, h^1, C^1, H^1) (the vector of random variables) and let $f_1 : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ be any function. Suppose that $f_1(\mathbf{Z}, \varepsilon, N) = f_1(\varepsilon, \mathfrak{b}^1(\mathbf{Z}, \varepsilon, N))$ is increasing in ε, and satisfies $[f_1(\mathbf{Z}, \varepsilon_0, N), \; f_1(\mathbf{Z}, \varepsilon_I, N)] = [a, b]$. Let $f_2 : \mathbb{R} \times \mathbb{R}^8 \to \mathbb{R}$ be any function. We define

$$\mathfrak{S}^1_{f_1, f_2}(\mathbf{Z}, \mathbf{z_0}, a, b, N) = \sum_{k=1}^{M} f_2(\varepsilon_k, (\mathfrak{b}^1, \Delta \mathfrak{b}^1)(\mathbf{Z}, \mathbf{z_0}, \varepsilon_k, N)) \Delta f_1(\varepsilon_k, \mathfrak{b}^1(\mathbf{Z}, \mathbf{z_0}, \varepsilon_k, N)). \tag{35}$$

where $\Delta \mathfrak{b}^1 = (\Delta c^1, \Delta h^1, C\Delta^1, \Delta H^1)$. It is clear that

$$\mathfrak{S}^1_{f_1, f_2}(\mathbf{Z}, \mathbf{z_0}, a, b, N) = \mathfrak{S}^1_{f_1, f_2}(\mathbf{X}, \mathbf{x_0}, a, b, N) \tag{36}$$

and that

$$\lim_{M \to \infty} \lim_{N \to \infty} \mathfrak{S}^1_{f_1, f_2}(\mathbf{Z}, \mathbf{z_0}, a, b, N) = \lim_{M \to \infty} E(\mathfrak{S}^1_{f_1, f_2}(\mathbf{Z}, \mathbf{z_0}, a, b, N)) = \mathfrak{I}^1_{f_1, f_2}(\mathbf{Z}, \mathbf{z_0}, a, b) \tag{37}$$

where

$$\mathfrak{I}^1_{f_1,f_2}(\mathbf{Z},\mathbf{z_0},a,b) = \int_{f_1(\varepsilon,\mathfrak{b}^1(\mathbf{Z},\mathbf{z_0},\varepsilon))\in[a,b]} f_2(\varepsilon,(\mathfrak{b}^1,\frac{\partial\mathfrak{b}^1}{\partial\varepsilon})(\mathbf{Z},\mathbf{z_0},\varepsilon))df_1(\varepsilon,\mathfrak{b}^1(\mathbf{Z},\mathbf{z_0},\varepsilon)) \tag{38}$$

and

$$g^1(\mathbf{Z},\mathbf{z_0},\varepsilon) = \lim_{N\to\infty} g^1(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = E(g^1(\mathbf{Z},\mathbf{z_0},\varepsilon,N)) \tag{39}$$

for $g^1 \in \mathfrak{b}^1$, so that U-statistics given by these sums are invariant under $\mathfrak{H}_G$.

In the case of order 2 statistics we shall use

$$\mathfrak{S}^2_{f_1,f_2}(\mathbf{Z},a,b,N) = \mathfrak{S}^2_{f_1,f_2}(\mathfrak{H}_G(\mathbf{Z}),a,b,N) \tag{40}$$

as a definition instead of as a result, where the definition of $\mathfrak{S}^2_{f_1,f_2}(\mathbf{X},a,b,N)$ is obtained by replacing the 1's corresponding to the order of the statistic to 2's, and the building block variables of order 2 are defined by

$$h^2(\mathbf{Z},\varepsilon,N) = \frac{2}{N(N-1)}\sum_{p<q} I(Z_i^p - Z_i^q,\varepsilon) \tag{41}$$

(using any $i = 1,...,m$) and

$$H^2(\mathbf{Z},\varepsilon,N) = \frac{2}{N(N-1)}\sum_{p<q} I(\max_{1\le i\le m}(Z_i^p - Z_i^q),\varepsilon) \tag{42}$$

(recall there is no $\mathbf{z_0}$).

We shall restrict our attention to homogenizations to the uniform and normal distributions. In some cases, nonlinear functions of several of these sums will be used. We shall refer to any of these functions as an integral U-statistic.

In the case of IID distributions $\mathbf{Z}$, homogenization to the uniform distribution is a natural operation, because then the support of the building block random variables is uniformly distributed on the m-cube $[0,1]^m$. that is, once ε is discretized, each of the building block functions is a sum of random variables of the form $I_C(\mathbf{Z})$, where

$$I_C(\mathbf{Z}) = \begin{cases} 1 & \mathbf{Z}\in C, \\ 0 & \mathbf{Z}\notin C, \end{cases} \tag{43}$$

and C is any of the cubes

$$C_{\mathbf{k}} = \{\mathbf{x} \in\mathbb{R}^m : (\mathbf{k}-\mathbf{1})\varepsilon\le \mathbf{x} \le \mathbf{k}\varepsilon, \mathbf{k}\in\{1,...,I\}^m\} \tag{44}$$

formed by the ε grid (here $\le$ holds for each entry, $\mathbf{1} = (1,...1)\in\mathbb{N}^m$). This implies by the law of large numbers that when k^m is not too small (say larger than 20) the building block random variables are approximately normal. However, the sample on these random variables is approximately of size $\frac{k^m}{I^m}N$, which reduces the values of k for which there is a reasonable sample as m increases.

Some particular integrals

For our Monte-Carlo study we defined and studied the statistics listed in Table 1. It is understood that in the case of order 2 integrals there is no dependence on $\mathbf{z_0}$. We write

$L^{p,q}[a,b]$ for L^p measures taken along the variable ε on the interval $[a,b]$ with measure $\left(\frac{dH^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)}{d\varepsilon}\right)^q d\varepsilon$. For $q=1$ the integrals involved in these functions correspond to choosing in our definition $f_1(\varepsilon,c,h,C,H)=H$, and function(s) $f_2(\varepsilon, c, h, C, H, \Delta c, \Delta h, \Delta C, \Delta H)$ given by $|C/c^m|^p$, $|C-c^m|^p$, $|Ln(C)/(mLn(c))|^p$, etc. More generally, we may write $f_1=\varepsilon$ and $f_2=|C/c^m|^p\,\Delta H^q$, etc.

In practice we calculate approximations of the integrals, obtained using the Riemann sum corresponding to a partition ε_k, and we choose some fixed a,b in $(0,1)$. Let us refer to these approximations as $\mathfrak{s}_i^j$, $i=1,...,16$.

Theorem 3 *Let Z_i be a strictly stationary process which is absolutely regular (as in Theorem 1). Generically, the statistics $\mathfrak{s}_i^j$, $i=1,...,17$, are asymptotically normal as $N\to\infty$. If $\mathbf{Z}$ are IID then the means are 0, 1 and $\|1\|_{L^{p,q}[a,b]}$ according to wether i is an element of the set*

$$\{1,2,3,4,6,7,9,10,13,14,15\},\{11,12\} \text{ or } \{5,8,16\}. \tag{45}$$

Proof: As before, $\mathfrak{S}_i^j(\mathbf{Z},\mathbf{z_0})$ are asymptotically normal because they are smooth functions of the building block statistics, which are themselves asymptotically normal. The means are obtained by replacing the component statistics with their means, and using $E(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))=E(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))^m$.■

The Monte-Carlo experiment

We carried out a Monte Carlo experiment to study some of the properties of the integral statistics. We included four sets of specific statistics.

The first, numbered 1 to 12, consists of statistics related to the SNT and BDS statistics. These are statistics $\mathfrak{S}_1^j$ to $\mathfrak{S}_4^j$ evaluated in triples using $\varepsilon=0.25, 0.5, 0.75$.

The second, numbered 13 to 28, consists of the integral statistics $\mathfrak{S}_5^j$ to $\mathfrak{S}_{12}^j$ by pairs, evaluated with $p=1$, $[a,b]=[0.02,0.98]$ and $q=1$ or 0.5. These are statistics based on $L^{p,q}$ norms of functions of $C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)$ and $c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m$).

The third, numbered 29 to 36, consists of the integral statistics $\mathfrak{S}_{13}^j$ to $\mathfrak{S}_{16}^j$ by pairs, evaluated with the same parameters p,a,b,q. These are statistics based on $L^{p,q}$ norms of functions of $\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)$ $\Delta c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m$). These integral statistics thus aren't double integrals, as $\mathfrak{S}_5^j$ to $\mathfrak{S}_{12}^j$ are if we consider $C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)$ which is an accumulated distribution, as an integral.

Recall that in the case of time series Z the random variable $\mathbf{Z}$ consists of m-histories. Let us write $\mathbf{Z}(m)$ for this set. Summarizing, these three sets of statistics are defined by

$$\begin{aligned}
\mathfrak{I}_{3(n-1)+k}^j(m,Z,N) &= \mathfrak{S}_n^j(\mathbf{Z}(m),\boldsymbol{\mu},0.25k,N),\ n=1,...,4;\ k=1,2,3 \\
\mathfrak{I}_{12+2(n-1)+k}^j(m,Z,N) &= \mathfrak{S}_n^j(\mathbf{Z}(m),\boldsymbol{\mu},N,1,0.02,0.98,1-0.5(k-1)), \\
&\qquad n=5,...,16;\ k=1,2.
\end{aligned}$$

The order one statistics are centered at the mean, i.e., $\mathbf{z}_0=\boldsymbol{\mu}$. The integrals are approximated by the Riemann sum given in their definition, with $a=0.02, b=0.98$. Δ

follows definition (34), with $\varepsilon_k = \frac{k}{255}$, $k = 0, 255$.

The fourth set of statistics, numbered 37 to 41 consists of five CDR regressions on the intervals $[0.02, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, $[0.8, 0.98]$. Recall that in the series homogenized to the uniform distribution, the CDR and $G\&P$ dimension regressions are equivalent.

Let us write $\mathfrak{T}_i^j(m, Z, N), i = 1, ..., 41$ for these statistics, for which a brief description is given in Table 2.

For purposes of comparability, we test the same time series Z and lengths $N = 380$ and 2000 that Barnett et. al. (1996) use in their double bind experiment. These were series obtained from specific logistic, GARCH, NLMA, ARCH and ARMA processes. The only difference is that we do not extract the linear structure from these series before applying the tests. Since in practice this procedure is only applied to the ARMA series, comparability is preserved in the remaining cases and we thus test a series containing only linear dependence. To these five series we add the normal and the uniform distributions.

The reshuffling test we use is based on the following proposition:

Proposition 4 *Let $X_0, X_1, ...X_M$ be IID random variables having an integrable distribution density. Then*

$$P(\#\{k : X_k < X_0\} \leq r) = \frac{r}{M+1}.$$

Proof: Write $P(X_k < x) = F(x) = \int_{-\infty}^{x} dF(x)$. Then F is bounded and

$$\begin{aligned} P(\#\{k : X_k < X_0\} = s) &= \int_{-\infty}^{\infty} \binom{M}{s} F(x)^s (1 - F(x))^{M-s} dF(x) \\ &= \binom{M}{s} \int_0^1 y^s (1-y)^{M-s} dy \end{aligned}$$

Let $I_s = \int_0^1 y^s (1-y)^{M-s} dy$. Integrating by parts

$$\begin{aligned} I_s &= \frac{1}{s+1} \left[y^{s+1} (1-y)^{M-s} \right]_0^1 + \frac{M-s}{s+1} \int_0^1 y^{s+1} (1-y)^{M-s-1} dy \\ &= \begin{cases} \frac{M-s}{s+1} I_{s+1} & 0 \leq s \leq M-1, \\ \frac{1}{M+1} & s = M. \end{cases} \end{aligned}$$

Hence, by induction,

$$I_s = \frac{(M-s)!}{(s+1)...M} \frac{1}{M+1} = \frac{(M-s)!s!}{(M+1)!}$$

so

$$P(\#\{k : X_k < X_0\} = s) = \frac{M!}{(M+1)!} = \frac{1}{M+1},$$

from which the result follows.■

We thus define the following reshuffling test. For any time series Z (recall this is a random variable) let Z^R, be a reshuffling of its terms (without replacement). For any statistical test $\mathfrak{T}$, let $X = \mathfrak{T}(Z)$, $X^R = \mathfrak{T}(Z^R)$.

Corollary 5 *If Z is IID then it belongs to the same distribution that any of its reshuffled versions. Therefore the probability that X is amongst the r smallest (similarly largest) results of M applications of $\mathfrak{T}$ to Z^R is $\frac{r}{M+1}$.* ■

Consider the distribution of the results $\mathfrak{T}(Z)$ as compared to $\mathfrak{T}(Z^R)$. As long as the accumulated frequency distributions of $\mathfrak{T}(Z^0)$ and $\mathfrak{T}(Z^p)$ are continuous, there exists a unique x for which $P(\mathfrak{T}(Z) \leq x) = P(\mathfrak{T}(Z^R) \geq x)$, because the first function increases (while the second decreases) monotonically from 0 to 1 (1 to 0 respectively). We define the number $\pi = \max[x, 1-x]$ as the power of the reshuffling test based on $\mathfrak{T}$ used to discriminate between time series Z and Z^R, when power equals size.

In the Monte-Carlo experiment we set $M = 999$, and record the 10 smallest (and largest) levels for the statistical tests $\mathfrak{I}_i^j$ applied to Z and to Z^R. Then we observe if the r^{th} largest (or smallest) result of $\mathfrak{T}$ applied to instances of the original series lies below (or above) the r^{th} smallest (or largest) result of $\mathfrak{T}$ applied to reshuffles, for $r \leq 10$. For such r, the size and the power of the corresponding reshuffling test are simultaneously better than or equal to $\frac{r}{1000}$. We also measure the 1% confidence level in standard deviations from the mean, to see how well these confidence levels approximate those which would be obtained from assuming that the test has a normal distribution, and calculate the mean, standard deviation, asymmetry and kurtosis of the distribution $\mathfrak{I}_i^j(m, Z^R, N)$.

The Monte-Carlo experiment was carried out for $\varepsilon_k = \frac{k}{255}$, $k = 0, 255$, as was mentioned, and for $m = 1, ..., 32$.

The results

We thread our analysis of the results about statistics first applying $\mathfrak{H}_{\mathfrak{U}}$, thus homogenizing Z to the uniform distribution. Recall that homogenization has the advantage that, given Z is IID, the distribution of any statistic $\mathfrak{T}(Z)$ is independent of the distribution of Z. Homogenization to the *uniform* distribution seems the most natural in the case of the statistics we are considering, and has the additional advantages that the building block random variables are approximately normal [see equations (43), (44)]. Thus one of our purpose is to argue that, at the very least, $\mathfrak{H}_{\mathfrak{U}}$ does not reduce the power of the statistics $\mathfrak{I}_i^j$, while indeed it may increase it.

We first analyze the distributions $\mathfrak{I}_i^j(m, Z^R, N)$ when $\mathfrak{H}_{\mathfrak{U}}$ is applied. Since for any Z, $\mathfrak{I}_i^j(m, Z^R, N) = \mathfrak{I}_i^j(m, \mathfrak{U}, N)$, where $\mathfrak{U}$ is a uniformly distributed time series, we need only analyze the distribution of $\mathfrak{I}_i^j(m, \mathfrak{U}, N)$. Let us write $C^j(m, Z, \varepsilon, N) = C^j(\mathbf{Z}(m), \boldsymbol{\mu}, \varepsilon, N)$. Recall that $C^j(m, Z^R, \varepsilon, N) = C^j(m, \mathfrak{U}, \varepsilon, N)$, since $\mathfrak{H}_{\mathfrak{U}}$ is applied. Besides recording the results for the statistics $\mathfrak{I}_i^j$, we recorded for each Z and N the values of k corresponding to $C^j(m, Z^R, \varepsilon_k, N) = x$ for $x \in \{0.02, 0.212, 0.404, 0.596, 0.788, 0.98\}$ (the dimension regression interval endpoints). Let us write $k_{m,x}$ for these values. The mean values of $k_{m,x}$ over 999 repetitions of $\mathfrak{U}$ at $N = 2000$ was an increasing function of m. In the case of order one statistics, the five functions $k_{m,x}$ remained distinct over the range $m = 1, ..., 32$, although $k_{32,0.02} \sim 225$ (recall that the uppermost value of the ε grid is $k = 255$). In the case of order 2 statistics, however,

$k_{m,0.02} = 255$ for $m \geq 15$, so that all of the tested statistics are trivial in this range. Thus in the case of order 2 we report our results only up to $m = 15$. Similar ranges hold for $N = 380$, although they display more variance. Indeed, it is well-known that such values of N are small for the types of statistics we are dealing, so almost all of our report will deal with $N = 2000$.

The first observation is that the distributions of $\mathfrak{I}_i^j(m, \mathfrak{U}, 2000)$ are not sufficiently close to the normal distribution to be able to draw inference at the 1% level. The graph of the 10th smallest and 10th largest values of the statistics $\mathfrak{I}_i^j(m, \mathfrak{U}, 2000)$ (measured in standard deviations from the mean) through the relevant dimensions differ considerably from the graph of the sensitive (constant) 1% and 99% confidence levels of the normal distribution. Graphs 1 and 2 shows the average of histograms for m ranging from 1 to 32 in the case of order 1 and 2 to 15 in the case of order 2 for the statistics $\mathfrak{I}_i^1(m, \mathfrak{U}, 2000)$.

The second observation is that while the uniform and normal distributions are usually well detected, this is not always the case. For each statistic $\mathfrak{I}_i^j(m, Z, N)$ let $\pi_i^j(m, Z, N)$ be the number π defined above (power when power equals size). We exclude from any further comparison results $\mathfrak{I}_i^j(m, Z, N)$ (when Z is one of the nonlinear series to be tested) for which $\pi_i^j(m, \mathfrak{U}, N)$ and $\pi_i^j(m, \mathfrak{N}, N)$ are not both greater then 0.01. Tables 3.j.1 (where j = 1 or 2 is the order of the statistic) show those dimensions for which $\mathfrak{U}$ and $\mathfrak{N}$ were both detected with at least 1% confidence in the case of tests based on the homogenization to the uniform distribution $\mathfrak{H}_{\mathfrak{U}}$, with series of 2000 terms. Tables 3.j.2 to 3.j.4 Show how these results differed when (a) $\mathfrak{H}_{\mathfrak{U}}$ was replaced with $\mathfrak{H}_{\mathfrak{N}}$; (b) no homogenization was applied; and (c) the length of the series was changed to 380. It can be observed that $\mathfrak{H}_{\mathfrak{U}}$ worked well, the main exceptions being, in order 1, the SNT with $\varepsilon = 0.75$, dimensions 5 to 16, and low to intermediate dimensions for integrals 29 to 36. In order 2 the exceptions were the BDS statistic for $\varepsilon = 0.25$, dimensions 5 to 15, an assortment of very low and relatively high dimensions for integrals 33 to 35, and the dimension regression on $[0.020, 0.212]$ for dimensions 2 to 7. Homogenization to the normal distribution represented a trade-off in order 1 (the SNT statistics showed better results), and got almost full marks in order 2. No homogenization got full marks in both orders. But then $\mathfrak{H}_{\mathfrak{U}}$ also got full marks in both orders at $N = 380$. What underlies these results is changes in the ranges of epsilon for which the tests best detect independence, which happen due to the change of variables in the data, which we believe have to be understood *per se*, rather than arriving at blanket conclusions on homogenization..

Previous work (Mayer, 1995; Mayer and Feliz, 1996) has been concerned with such issues as wether the dimension readings are biased, producing spuriously 'low dimensional' results. Here we shall only be concerned with the power of the tests. We define an index P of the power of statistics in the following manner. For each statistic $\mathfrak{I}_i^j(m, Z, 2000)$ let $\pi_i^j(m, Z, 2000)$ be the number π defined above (power when power equals size). Then

$$P_{i,a \text{ to } b}^j(Z, \mathfrak{H}) = \sum_{m=a}^{b} \Theta(\pi_i^j(m, Z, 2000)),$$

where $\mathfrak{H}$ is the homogenization process used, i.e., $\mathfrak{H}_{\mathfrak{U}}$, $\mathfrak{H}_{\mathfrak{N}}$ or the identity transformation $\mathfrak{Id}$, (which has been omitted from the notation in the right hand side all along for convenience, and

$$\Theta(\pi) = \begin{cases} \frac{1}{1000}\pi^{-1} & \text{if } \pi \leq 0.01, \\ 0 & \text{if } \pi > 0.01. \end{cases}$$

Thus, for example, Θ is 1 for a high value of power $\pi = 0.001$ (the highest possible in this test), and 0.1 for a minimally acceptable level of power $\pi = 0.01$, and is zero for lower levels of power. The power index is the sum of these over the ranges of dimensions 2 to 8 and 9 to 32 (or 9 to 15), for order 1 (order 2) statistics respectively. We chose these ranges to separate out the high dimensional results, because of the specific difficulties that work in high dimensions involve (Ramsey, Sayers and Rothman, 1990). Thus we shall be concerned with the power indices $P^1_{i,2\text{ to }8}$, $P^1_{i,9\text{ to }32}$, $P^2_{i,2\text{ to }8}$, $P^2_{i,9\text{ to }15}$ for the statistical tests $i = 1$ to 41. We aggregate these power indices in several ways to compare the performance of the homogenization options, the different integral tests, and to see which tests were best for each non-linear series.

Tables 4.j show which of the homogenization options ($\mathfrak{H}_{\mathfrak{U}}$, $\mathfrak{H}_{\mathfrak{N}}$, or none) obtained the maximum index $\frac{1}{b-a+1}\sum_{i=1}^{41} P^1_{i,a\text{ to }b}$ for the relevant dimension ranges a to b for $N = 2000$. By this measure Order 2 statistics almost strictly dominate order 1 statistics, and homogenization to the uniform distribution often produces the best results. The higher dimension and lower dimensional ranges are comparable in their results.

The ARMA and logistic structures were easy to detect. Table 5 show what percentage of the statistics in each of the ranges 1 to 12 (SNT and BDS type), 13 to 28 ($L_{p,q}$ norms of $C^j(m, Z, \varepsilon, N)$), 29 to 36 ($L_{p,q}$ norms of $\Delta C^j(m, Z, \varepsilon, N)$), and 37 to 41 (dimensions regressions 1 to 5) detected these structures at the highest levels of $P^j_{i,a\text{ to }b}$. For $N = 2000$, integral range 13 to 28 with $\mathfrak{H}_{\mathfrak{U}}$ always obtained the best scores. The integral range 29 to 36 with no homogenization obtained some ties, as did the BDS statistic in the case of the logistic. The dimension measures came close to a tie in the case of the logistic for the high dimensional range of in order 1.

In Table 6, it may be observed that ARCH and NLMA were best detected in the higher dimensional ranges (for both orders), by integrals 13 to 16 in the order 1 case and by the first dimension regression (Dim 1) in order 2. Both of these obtained power levels of 0.001 for several distinct dimensions. GARCH was best detected at the lower dimensional range, by integrals 27, 13 and 14 in order 1 and 32, 29-30 and 36 in order 2.

Which were the best tests? Since the SNT and BDS statistics are well-known and have been used extensively, we take them as a bench-mark for purposes of comparison. This time we use the index

$$Q^j_{i,a\text{ to }b}(Z) = \max_{\mathfrak{H}\in\{\mathfrak{H}_{\mathfrak{U}},\mathfrak{H}_{\mathfrak{N}},\mathfrak{Id}\}} P^j_{i,a\text{ to }b}(Z, \mathfrak{H})$$

obtained by maximizing $P^j_{i,a\text{ to }b}$ over the normalization options for $N = 2000$, thus comparing the best performance of each statistic. We first deal with statistical tests 1 to 12. These represent four functional modifications of the SNT and BDS tests, applied for three values of ε. Although these tests are very similar in their performance, in the case of the order 1 tests $\mathfrak{S}^j_4$ is dominant for each value of ε (getting better than or equal results to the other tests including the SNT); while in order 2 it is almost always dominant. The exceptions occur for $\varepsilon = 0.5$ or 0.75. In all but one of these $\mathfrak{S}_3^j$ is dominant, while in the remaining case $\mathfrak{S}_2^j$ is. The results of the BDS statistic are always equal or worse to one of these statistics.

To evaluate the other tests we compared them to the maximum performance index of tests 1-12. There weren'tt any integral tests which strictly dominated the SNT and BDS statistics for all of the time series tested. However, there were integrals whose average performance was better. This average $\overline{Q}^j_{i,a\text{ to }b}$ was calculated as follows

$$\begin{aligned}\widehat{Q}^j_{i,a\text{ to }b}(Z) &= Q^j_{i,a\text{ to }b}(Z)\left[\max_{i=1,\ldots,41} Q^j_{i,a\text{ to }b}(Z)\right]^{-1},\\ \overline{Q}^j_{i,a\text{ to }b} &= \tfrac{1}{5}\sum_{\substack{Z\in\{\text{ARCH, ARMA, GARCH,}\\ \text{NLMA, Logistic}\}}} \widehat{Q}^j_{i,a\text{ to }b}(Z),\end{aligned}$$

so that performance for each time series was measured relative to the best performance obtained in detecting its own type of dependence. By this measure, Table 7 shows the statistics which scored better than the SNT or BDS statistics:

Tables 8 and 9 show the results for the individual statistics, averaged separately for the non-linear stochastic time series (ARCH, GARCH, NLMA), which were harder to detect, and for the linear stochastic and deterministic time series (ARMA and Logistic), which were easier to detect.

Conclusions

Our contributions cover two main aspects. The first is using homogenization before applying U-statistics. In this respect our conclusion is that, although there are trade-offs and sometimes one option is better than another, on the whole homogenization actually increases the power of these statistics. Besides, the trade-offs have really to do with which ranges of epsilon are best to use, rather than with losses due to homogenization itself. In this respect further study is still needed. However, the door is definitely open for the consideration of universal confidence intervals. These are especially relevant in that the distributions characterizing the tests are quite clearly not sufficiently close to the normal distribution for drawing inference.

The second contribution is using integral statistics so as to use the information available from a wider range of epsilon.. In this respect, we have found single tests which perform better than the whole class of SNT or BDS statistics and values of epsilon tested (integrals 1 to 12) over the low or over the high dimension ranges. Nevertheless, the results are in that there are more functional forms which can be considered and which may give improved results. Again, this is also a matter of understanding which ranges of epsilon are most relevant, perhaps to specific processes.

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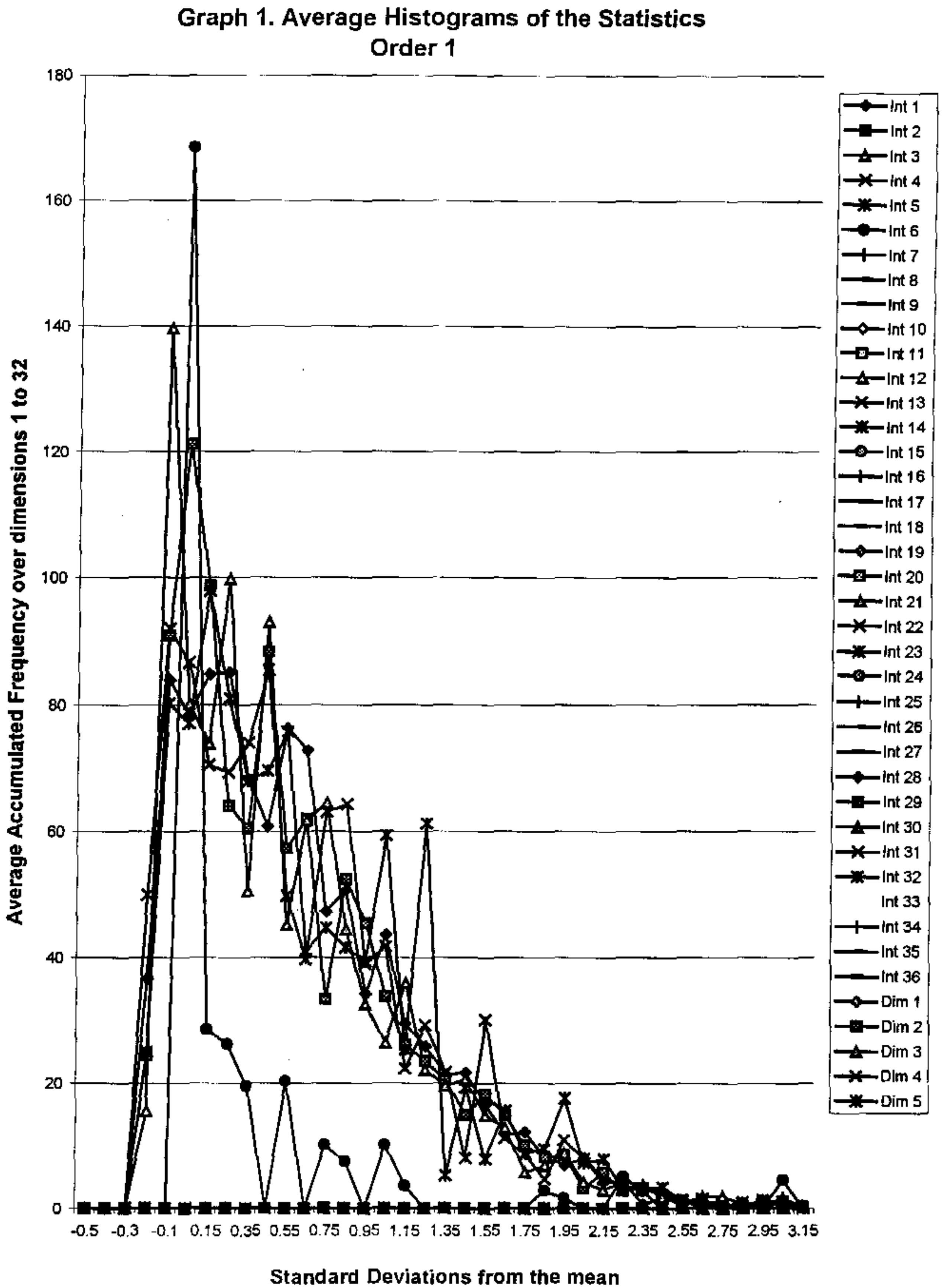
$\mathfrak{S}_1^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m$ (SNT and BDS statistics)
$\mathfrak{S}_2^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)/c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m - 1$
$\mathfrak{S}_3^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = \ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)) - m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))$
$\mathfrak{S}_4^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = \ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))/\left[m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\right] - 1$
$\mathfrak{S}_5^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)/c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_6^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)/c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m - 1\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_7^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_8^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))/\left[m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\right]\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_9^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))/\left[m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\right] - 1\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{10}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)) - m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{11}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)\Vert_{L^p} / \Vert c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{12}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\Vert_{L^p} / \Vert m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{13}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert[\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - \mu]/\sigma_1\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{14}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert[\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - \mu]/\sigma_2\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{15}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - \Delta(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m)\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{16}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)/\Delta(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m)\Vert_{L^{p,q}[a,b]}$

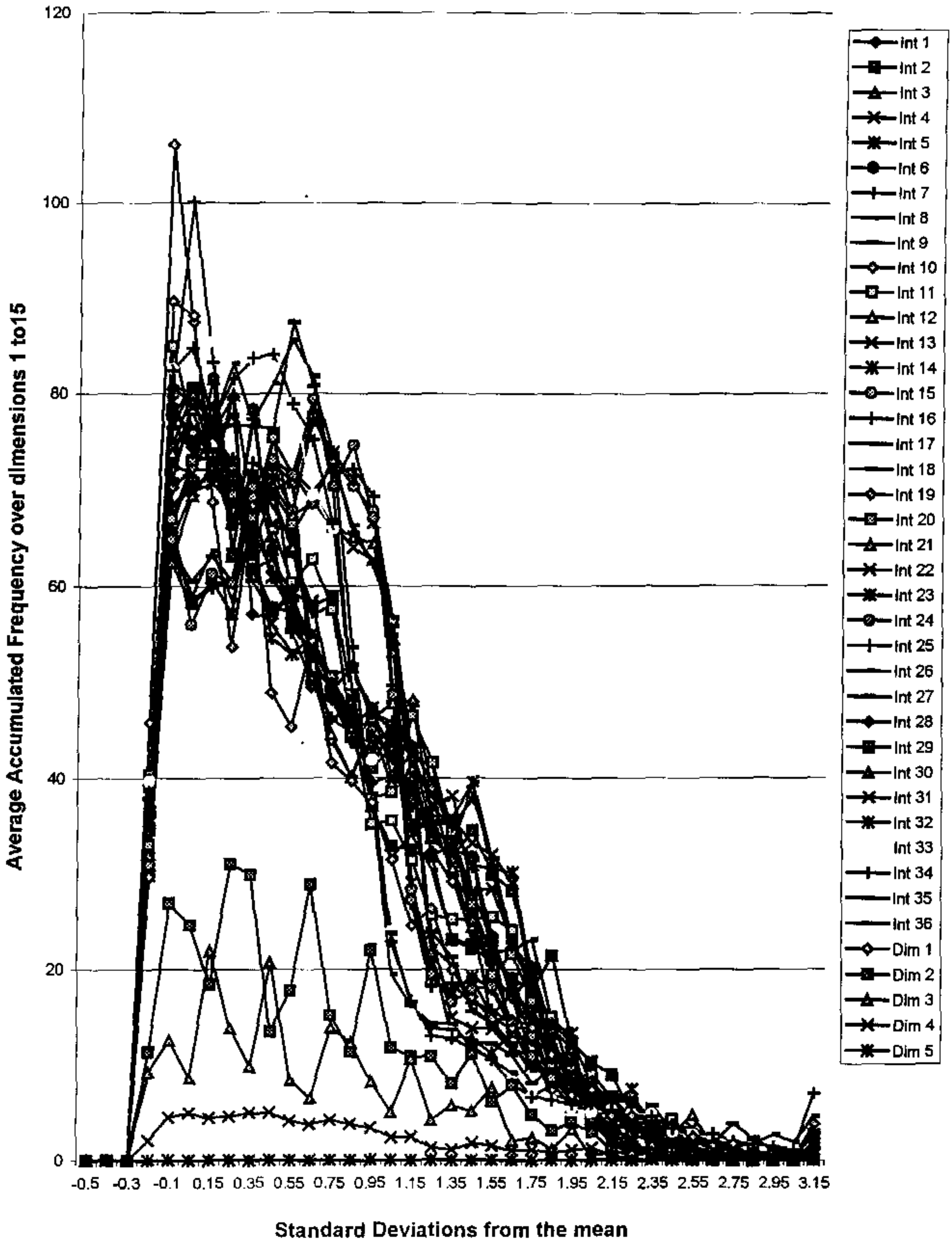
Table 1. Definition of the Statistics of Order j used in the Monte-Carlo tests

$\mathfrak{S}_2^j, \mathfrak{S}_3^j, \mathfrak{S}_4^j$ are functional modifications of the SNT and BDS statistics $\mathfrak{S}_1^j$. The other statistics are integral statistics.

μ is the theoretical mean of the distribution $\Delta(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m)$, and $\sigma_1 = \frac{1}{2}|\mu|^{1/2}$, $\sigma_2 = \frac{1}{2}|\mu(1-\mu)|^{1/2}$. are two approximations of its standard deviation. The idea behind the definition of $\mathfrak{S}_{13}^j, \mathfrak{S}_{14}^j$ is that these consist of sums of random variables approximating the standard normal distribution, one for each element of the ε grid. Each of $\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon_k,N)$ is approximately normal when $\mathbf{Z}$ is IID and homogenization to the uniform distribution is used, because it is the sum of $k^m - (k-1)^{m-1}$ identical uniform distributions $I_C(\mathbf{Z})$, [see (43), (44)]. However, the sample for each k is approximately $\frac{k^m-(k-1)^{m-1}}{I^m}N$.


Graph 1. Average Histograms of the Statistics
Order 1
Average Accumulated Frequency over dimensions 1 to 32
Standard Deviations from the mean
0
20
40
60
80
100
120
140
160
180
-0.5 -0.3 -0.1 0.15 0.35 0.55 0.75 0.95 1.15 1.35 1.55 1.75 1.95 2.15 2.35 2.55 2.75 2.95 3.15
Int 1
Int 2
Int 3
Int 4
Int 5
Int 6
Int 7
Int 8
Int 9
Int 10
Int 11
Int 12
Int 13
Int 14
Int 15
Int 16
Int 17
Int 18
Int 19
Int 20
Int 21
Int 22
Int 23
Int 24
Int 25
Int 26
Int 27
Int 28
Int 29
Int 30
Int 31
Int 32
Int 33
Int 34
Int 35
Int 36
Dim 1
Dim 2
Dim 3
Dim 4
Dim 5

Graph 2. Average Histograms of the Statistics Order 2



$\mathfrak{S}_1^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m$ (SNT and BDS statistics)
$\mathfrak{S}_2^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)/c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m - 1$
$\mathfrak{S}_3^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = \ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)) - m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))$
$\mathfrak{S}_4^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) = \ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))/\left[m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\right] - 1$
$\mathfrak{S}_5^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)/c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_6^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)/c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m - 1\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_7^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_8^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))/\left[m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\right]\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_9^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))/\left[m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\right] - 1\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{10}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)) - m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{11}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)\Vert_{L^p} / \Vert c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{12}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\ln(C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\Vert_{L^p} / \Vert m\ln(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N))\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{13}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert[\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - \mu]/\sigma_1\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{14}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert[\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - \mu]/\sigma_2\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{15}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N) - \Delta(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m)\Vert_{L^{p,q}[a,b]}$
$\mathfrak{S}_{16}^j(\mathbf{Z},\mathbf{z_0},N,p,a,b,q) = \Vert\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)/\Delta(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m)\Vert_{L^{p,q}[a,b]}$

Table 1. Definition of the Statistics of Order j used in the Monte-Carlo tests

$\mathfrak{S}_2^j$, $\mathfrak{S}_3^j$, $\mathfrak{S}_4^j$ are functional modifications of the SNT and BDS statistics $\mathfrak{S}_1^j$. The other statistics are integral statistics. μ is the theoretical mean of the distribution $\Delta(c^j(\mathbf{Z},\mathbf{z_0},\varepsilon,N)^m)$, and $\sigma_1 = \frac{1}{2}|\mu|^{1/2}$, $\sigma_2 = \frac{1}{2}|\mu(1-\mu)|^{1/2}$. are two approximations of its standard deviation. The idea behind the definition of $\mathfrak{S}_{13}^j$, $\mathfrak{S}_{14}^j$ is that these consist of sums of random variables approximating the standard normal distribution, one for each element of the ε grid. Each of $\Delta C^j(\mathbf{Z},\mathbf{z_0},\varepsilon_k,N)$ is approximately normal when $\mathbf{Z}$ is IID and homogenization to the uniform distribution is used, because it is the sum of $k^m - (k-1)^{m-1}$ identical uniform distributions $I_C(\mathbf{Z})$, [see (43), (44)]. However, the sample for each k is approximately $\frac{k^m-(k-1)^{m-1}}{l^m}N$.

Tests applied in Monte-Carlo experiment

Test	Brief description of test
Integral 1	$C(m, \varepsilon) - C(1, \varepsilon)^m$, $\varepsilon = 0.25$ (SNT or DBS)
Integral 2	$C(m, \varepsilon) - C(1, \varepsilon)^m$, $\varepsilon = 0.50$ (SNT or DBS)
Integral 3	$C(m, \varepsilon) - C(1, \varepsilon)^m$, $\varepsilon = 0.75$ (SNT or DBS)
Integral 4	$C(m, \varepsilon)/C(1, \varepsilon)^m$, $\varepsilon = 0.25$
Integral 5	$C(m, \varepsilon)/C(1, \varepsilon)^m$, $\varepsilon = 0.50$
Integral 6	$C(m, \varepsilon)/C(1, \varepsilon)^m$, $\varepsilon = 0.75$
Integral 7	$\ln C(m, \varepsilon) - m \ln C(1, \varepsilon)$, $\varepsilon = 0.25$
Integral 8	$\ln C(m, \varepsilon) - m \ln C(1, \varepsilon)$, $\varepsilon = 0.50$
Integral 9	$\ln C(m, \varepsilon) - m \ln C(1, \varepsilon)$, $\varepsilon = 0.75$
Integral 10	$\ln C(m, \varepsilon)/(m \ln C(1, \varepsilon)) -1$, $\varepsilon = 0.25$
Integral 11	$\ln C(m, \varepsilon)/(m \ln C(1, \varepsilon)) -1$, $\varepsilon = 0.50$
Integral 12	$\ln C(m, \varepsilon)/(m \ln C(1, \varepsilon)) -1$, $\varepsilon = 0.75$
Integral 13	$\|C(m, \varepsilon)/C(1, \varepsilon)^m\|_{p,q}$[‡] $q = 1.0$
Integral 14	$\|C(m, \varepsilon)/C(1, \varepsilon)^m\|_{p,q}$ $q = 0.5$
Integral 15	$\|1 - C(m, \varepsilon)/C(1, \varepsilon)^m\|_{p,q}$ $q = 1.0$
Integral 16	$\|1 - C(m, \varepsilon)/C(1, \varepsilon)^m\|_{p,q}$ $q = 0.5$
Integral 17	$\|C(m, \varepsilon) - C(1, \varepsilon)^m\|_{p,q}$ $q = 1.0$
Integral 18	$\|C(m, \varepsilon) - C(1, \varepsilon)^m\|_{p,q}$ $q = 0.5$
Integral 19	$\|\ln C(m, \varepsilon)/(m \ln C(1, \varepsilon))\|_{p,q}$ $q = 1.0$
Integral 20	$\|\ln C(m, \varepsilon)/(m \ln C(1, \varepsilon))\|_{p,q}$ $q = 0.5$
Integral 21	$\|1 - \ln C(m, \varepsilon)/(m\ln C(1, \varepsilon))\|_{p,q}$ $q = 1.0$
Integral 22	$\|1 - \ln C(m, \varepsilon)/(m \ln C(1, \varepsilon))\|_{p,q}$ $q = 0.5$
Integral 23	$\|\ln C(m, \varepsilon) - m \ln C(1, \varepsilon))\|_{p,q}$ $q = 1.0$
Integral 24	$\|\ln C(m, \varepsilon) - m \ln C(1, \varepsilon))\|_{p,q}$ $q = 0.5$
Integral 25	$\|C(m, \varepsilon)\|_{p,q}/\|C(1, \varepsilon)^m\|_{p,q}$ $q = 1.0$
Integral 26	$\|C(m, \varepsilon)\|_{p,q}/\|C(1, \varepsilon)^m\|_{p,q}$ $q = 0.5$
Integral 27	$\|\ln C(m, \varepsilon)\|_{p,q}/\|m \ln C(1, \varepsilon)\|_{p,q}$ $q = 1.0$
Integral 28	$\|C(m)\|_{p,q}/\|m \ln C(1, \varepsilon)\|_{p,q}$ $q = 0.5$
Integral 29	$\|(dC(m, \varepsilon) - \mu) / \sigma_1\|_{p,q}$ $q = 1.0$
Integral 30	$\|(dC(m, \varepsilon) - \mu) / \sigma_1\|_{p,q}$ $q = 0.5$
Integral 31	$\|(dC(m, \varepsilon) - \mu) / \sigma_2\|_{p,q}$ $q = 1.0$
Integral 32	$\|(dC(m, \varepsilon) - \mu) / \sigma_2\|_{p,q}$ $q = 0.5$
Integral 33	$\|dC(m, \varepsilon) - d(C(1, \varepsilon)^m)\|_{p,q}$ $q = 1.0$
Integral 34	$\|dC(m, \varepsilon) - d(C(1, \varepsilon)^m)\|_{p,q}$ $q = 0.5$
Integral 35	$\|1 - dC(m, \varepsilon)/d(C(1, \varepsilon)^m)\|_{p,q}$ $q = 1.0$
Integral 36	$\|1 - dC(m, \varepsilon)/d(C(1, \varepsilon)^m)\|_{p,q}$ $q = 0.5$
Integral 37 (Dim 1)	Regression1: $C(m, \varepsilon) \in [0.020, 0.212]$
Integral 38 (Dim 2)	Regression2: $C(m, \varepsilon) \in [0.212, 0.404]$
Integral 39 (Dim 3)	Regression3: $C(m, \varepsilon) \in [0.404, 0.596]$
Integral 40 (Dim 4)	Regression4: $C(m, \varepsilon) \in [0.596, 0.788]$
Integral 41 (Dim 5)	Regression5: $C(m, \varepsilon) \in [0.788, 0.980]$

Table 2

[‡] Denotes an L^p norm with integration measure $(\Delta C(m,\varepsilon))^q$.

Dimensions for which statistic obtained better than 1% power and size for Uniform and Normally distributed data (1 versus 0); Series Uniformized to Uniform distribution, 2000 terms
Order 1

Dimension	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Integral 1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 2	1	1	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 5	1	1	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 6	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 7	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 8	1	1	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 9	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 11	1	1	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 12	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 13	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 14	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 24	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 26	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 27	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 28	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 29	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 30	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 31	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 32	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 33	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 34	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 35	0	0	0	0	0	0	0	0	0	0	-0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 36	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 3.1.1

Dimensions for which statistic obtained better than 1% power and size for Uniform and Normally distributed data (1 versus 0); Homogeneization to Uniform distribution minus Homogeneization to Normal distribution, 2000 terms, Order 1

Dimension	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Integral 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 2	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 3	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 5	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 6	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 8	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 9	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 11	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 12	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 13	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 14	0	1	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 15	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 16	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 18	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 19	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 21	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 23	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 24	0	1	0	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 25	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 26	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 28	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 29	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 30	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 31	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 32	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 33	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 34	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 35	0	-1	-1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 36	0	-1	-1	-1	-1	0	0	0	-1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 1	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 2	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.1.2

Dimensions for which statistic obtained better than 1% power and size for Uniform and Normally distributed data (1 versus 0); Homogeneization to Uniform distribution minus no Homogeneization, 2000 terms, Order 1

Dimension	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Integral 1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 2	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 3	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 4	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 5	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 6	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 7	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 8	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 9	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 10	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 11	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 12	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 14	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 19	0	0	1	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 21	0	0	1	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 23	0	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 24	0	1	0	1	0	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 27	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 28	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 29	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	0	0	1	1	0	0	1	1
Integral 30	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	0	0	1	1	0	0	1	1
Integral 31	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
Integral 32	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	0	0	1	1	1	1	1	1	0	0	1	1	0	0	1	1
Integral 33	-1	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 34	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 35	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 36	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.1.3

Dimensions for which statistic obtained better than 1% power and size for Uniform and Normally distributed data (1 versus 0); Homogeneization to Uniform distribution, Case of 2000 terms minus case of 380 terms, Order 1

Dimension	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Integral 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 2	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 3	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 5	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 6	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 8	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 9	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 11	0	0	-1	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 12	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 14	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 24	0	0	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 28	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 31	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 32	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 33	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 34	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 35	0	0	0	0	0	0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 36	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 1	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.1.4

Dimensions for which statistic obtained better than 1% power and size for Uniform and Normally distributed data;
Series Uniformized to Uniform distribution, 2000 terms
Order 2

Dimension	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Integral 1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
Integral 2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 3	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 4	1	1	1	0	0	0	0	0	0	0	0	0	0	0
Integral 5	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 6	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 7	1	1	1	0	0	0	0	0	0	0	0	0	0	0
Integral 8	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 9	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 10	1	1	1	0	0	0	0	0	0	0	0	0	0	0
Integral 11	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 12	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 13	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 14	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 15	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 16	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 17	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 18	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 19	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 20	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 21	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 22	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 23	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 24	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 25	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 26	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 27	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 28	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 29	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 30	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 31	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 32	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Integral 33	0	0	1	1	1	1	0	0	0	0	0	0	0	0
Integral 34	0	0	1	1	1	1	1	0	0	0	0	0	0	0
Integral 35	0	1	1	1	1	1	1	1	1	1	0	0	0	0
Integral 36	0	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 1	1	0	0	0	0	0	1	1	1	1	1	1	1	1
Dim 2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 3	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 4	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Dim 5	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 3.2.1

Dimensions for which statistic obtained better than 1% power and size for Uniform and Normally distributed data (1 versus 0); Homogeneization to Uniform distribution minus Homogeneization to Normal distribution, 2000 terms, Order 2

Dimension	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Integral 1	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
Integral 2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 4	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
Integral 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 7	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
Integral 8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 10	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
Integral 11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 14	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 15	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 16	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 18	0	0	1	0	0	0	0	0	0	0	0	0	0	0
Integral 19	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 20	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 21	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 22	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 24	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 25	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 26	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 27	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 28	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 29	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 30	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 31	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 32	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 33	-1	-1	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
Integral 34	-1	-1	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1
Integral 35	-1	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1
Integral 36	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 1	0	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0
Dim 2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.2.2

Dimensions for which statistic obtained better than 1% power and size for Uniform and Normally distributed data (1 versus 0); Homogeneization to Uniform distribution minus no Homogeneization, 2000 terms, Order 2

Dimension	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Integral 1	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0
Integral 2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 4	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0
Integral 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 7	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0
Integral 8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 10	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0
Integral 11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 14	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 15	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 16	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 18	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 19	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 20	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 21	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 22	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 24	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 25	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 26	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 27	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 28	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 29	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 30	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 31	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 32	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 33	-1	-1	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
Integral 34	-1	-1	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1
Integral 35	-1	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1
Integral 36	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 1	0	-1	0	0	0	0	1	1	0	0	0	0	0	0
Dim 2	0	0	0	0	1	1	0	0	0	0	0	0	0	0
Dim 3	0	0	1	0	0	0	0	0	0	0	0	0	0	0
Dim 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.2.3

Dimensions for which statistic obtained better than 1% power and size for Uniform and Normally distributed data (1 versus 0); Homogeneization to Uniform distribution, Case of 2000 terms minus case of 380 terms,Order 2

Dimension	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Integral 1	0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0
Integral 2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 4	0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0
Integral 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 7	0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0
Integral 8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 10	0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0
Integral 11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 14	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 15	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 16	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 18	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 19	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 20	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 21	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 22	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 24	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 25	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 26	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 27	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 28	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 29	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 30	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 31	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 32	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Integral 33	-1	-1	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1
Integral 34	-1	-1	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1
Integral 35	-1	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1
Integral 36	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 1	0	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0
Dim 2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Dim 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.2.4

Comparison of Homogeneization methods (amongst Uni_{2000}, Nor_{2000}, No_{2000})

Homogeneization Method Obtaining Maximum Power Index $\frac{1}{b-a+1}\sum_{i=1}^{41} P_{i,a \text{ to } b}^{j}$

Order 1

Dimensions	ARCH	P	ARMA	P	GARCH	P	NLMA	P	Logistic	P
2-8	Uni_{2000}	0.1	Uni_{2000}*	24.9	Uni_{2000}	3.7	Nor_{2000}	0.5	Uni_{2000}	18.8
9-32	No_{2000}	1.1	Uni_{2000}	25.8	No_{2000}	0.7	No_{2000}	2.1	Uni_{2000}	11.5

Table 4.1

Homogeneization Method Obtaining Maximum Power Index $\frac{1}{b-a+1}\sum_{i=1}^{41} P_{i,a \text{ to } b}^{j}$

Order 2

Dimensions	ARCH	P	ARMA	P	GARCH	P	NLMA	P	Logistic	P
2-8	Uni_{2000}	1.6	Uni_{2000}*	36.5	Uni_{2000}	12.5	Uni_{2000}	1.1	No_{2000}*	33.0
9-15	Uni_{2000}	1.4	No_{2000}**	32.6	Uni_{2000}	1.8	Uni_{2000}	1.7	No_{2000}	29.9

Table 4.2

Uni_{2000}	Homogeneization to the Uniform Distribution, N = 2000.
Nor_{2000}	Homogeneization to the Normal Distribution, N = 2000.
No_{2000}	No homogeneization, N = 2000.
Uni_{380}	Homogeneization to the Uniform Distribution, N = 380.

* Uni_{380} obtained a higher score.

** Uni_{380} obtained a higher score, and Uni_{2000} is almost identical.

Statistical tests best detecting given non-linear time series

Order and Range of Dimensions	Average Number of Integrals Abov High Score	ARMA	Average Number of Integrals Above High Score	Logistic
Order 1 2-8	Homogeneization	**Uni$_{2000}$**	Homogeneization	**Uni$_{2000}$**
	Integral range	**13-28**	Integral range	**13-28**
	% > 6.9	**81**	% > 6.9	**81**
Order 1 9-32	Homogeneization	***Uni$_{2000}$, No$_{2000}$***	Homogeneization	Uni$_{2000}$
	Integral range	***13-28, 29-36***	Integral range	13-28, 37-41
	% > 23.9	***100***	% > 20	19[†], 20
Order 2 2-8	Homogeneization	***Uni$_{2000}$, No$_{2000}$***	Homogeneization	***Uni$_{2000}$, No$_{2000}$, No$_{2000}$***
	Integral range	***13-28, 29-36***	Integral range	***13-28, 29-36, 1-12***
	% > 6.9	***100***		***100***
Order 2 9-15	Homogeneization	***Uni$_{2000}$***	Homogeneization	***Uni$_{2000}$, No$_{2000}$***
	Integral range	***13-28***	Integral range	***13-28, 29-36***
	% > 6.9	***100***	% > 6.9	***100***

Table 5. Integral types best detecting ARMA and Logistic at 2000 terms (including ties and close calls). Percentage of integrals obtaining high score in given dimension range. Scores over 80% in bold. 100% also in italics.

Order and Range of Dimensions	Maximum Scores	ARCH	GARCH	NLMA
Order 1 2-8	Homogeneization	Uni$_{2000}$	**Uni$_{2000}$**	Nor$_{2000}$
	Integral	1, 4, 7, 10	**27, 13-14**	36
	Score	0.2	**4.3, 4.1**	0.7
Order 1 9-32	Homogeneization	**No$_{2000}$**	No$_{2000}$	**No$_{2000}$**
	Integral	**13 - 16**	1, 4, 7, 10	**13-16**
	Score	**6.5**	2.4	**8.6**
Order 2 2-8	Homogeneization	Uni$_{2000}$	**Uni$_{2000}$**	Uni$_{2000}$
	Integral	35, 36	**32, 29-30, 36**	1, 4, 7, 10
	Score	2.3, 2.2	**6.2, 6.0,6.0**	1.0
Order 2 9-15	Homogeneization	**No$_{2000}$**	Nor$_{2000}$	**No$_{2000}$**
	Integral	**Dim 1**	18	**Dim 1**
	Score	**6.0**	2.7	**5.5**

Table 6. Integral types best detecting ARCH, GARCH and NLMA at 2000 terms (including ties and close calls). Maximum Score obtained in given dimension range. Best scores in bold.

[†] This entry represents a score of 23.3

Dimensions	Order 1	Order 2
2-8	27, 13, 14, 26, 36, Dim 2, 15, 16, 25, 18, 28, 20, 23, 17, 19, 21, 22, 31, 32, 34, Dim 3.	36, Dim 1.
9-32 or 15	13, 15, 14, 16, 18, 28.	Dim 1, 18.

Table 7. Tests having weighted power index $Q^{j}_{i,a\,\mathrm{to}\,b}$ better than all tests 1 to 12, in order of performance.

Test	Order 1		Order 2	
	Dims 2-8	**Dims 9-32**	**Dims 2-8**	**Dims 9-15**
Integral 1	0.43	0.40	0.00	0.13
Integral 2	0.07	0.20	**0.64**	0.31
Integral 3	0.12	0.35	0.13	0.19
Integral 4	0.43	0.40	0.22	0.13
Integral 5	0.07	0.20	**0.64**	0.31
Integral 6	0.12	0.35	0.13	0.19
Integral 7	0.43	0.40	0.22	0.13
Integral 8	0.07	0.20	**0.64**	0.31
Integral 9	0.12	0.35	0.13	0.22
Integral 10	0.43	0.40	0.33	0.13
Integral 11	0.07	0.20	**0.64**	0.31
Integral 12	0.12	0.35	0.11	0.18
Integral 13	0.36	**0.69**	0.19	0.01
Integral 14	0.36	**0.69**	0.25	0.01
Integral 15	0.12	**0.69**	0.30	0.01
Integral 16	0.12	**0.69**	0.11	0.01
Integral 17	0.05	0.04	0.13	0.01
Integral 18	0.08	0.32	0.03	0.34
Integral 19	0.05	0.04	0.06	0.01
Integral 20	0.07	0.04	0.03	0.01
Integral 21	0.05	0.04	0.04	0.01
Integral 22	0.05	0.04	0.03	0.01
Integral 23	0.06	0.04	0.03	0.01
Integral 24	0.04	0.04	0.11	0.01
Integral 25	0.12	0.04	0.15	0.01
Integral 26	0.23	0.04	0.08	0.01
Integral 27	0.37	0.04	0.15	0.01
Integral 28	0.18	0.09	0.26	0.05
Integral 29	0.05	0.04	0.28	0.01
Integral 30	0.05	0.04	0.35	0.01
Integral 31	0.05	0.04	0.35	0.02
Integral 32	0.05	0.04	0.36	0.01
Integral 33	0.05	0.04	0.36	0.01
Integral 34	0.05	0.04	0.45	0.01
Integral 35	0.01	0.03	0.47	0.05
Integral 36	**0.50**	0.06	**0.69**	0.07
Integral 37 (Dim 1)	0.06	0.08	**0.67**	**0.96**
Integral 38 (Dim 2)	0.25	0.04	0.13	0.01
Integral 39 (Dim 3)	0.05	0.04	0.11	0.01
Integral 40 (Dim 4)	0.05	0.00	0.03	0.00
Integral 41 (Dim 5)	0.05	0.00	0.00	0.00

Table 8. Average power index $\frac{1}{3}\sum_{Z\in\{\mathrm{ARCH,GARCH,NLMA}\}}\hat{Q}^{j}_{i,a\,\mathrm{to}\,b}(Z)$; scores over 0.5 in bold ($N = 2000$).

Test	Order 1		Order 2	
	Dims 2-8	**Dims 9-32**	**Dims 2-8**	**Dims 9-15**
Integral 1	0.22	0.10	0.00	0.94
Integral 2	0.66	0.28	**1.00**	**1.00**
Integral 3	0.74	0.54	**1.00**	**1.00**
Integral 4	0.22	0.10	**1.00**	0.94
Integral 5	0.66	0.28	**1.00**	**1.00**
Integral 6	0.74	0.54	**1.00**	**1.00**
Integral 7	0.18	0.10	**1.00**	0.94
Integral 8	0.66	0.28	**1.00**	**1.00**
Integral 9	0.74	0.54	**1.00**	**1.00**
Integral 10	0.25	0.17	**1.00**	0.94
Integral 11	0.66	0.45	**1.00**	**1.00**
Integral 12	0.74	0.54	**1.00**	**1.00**
Integral 13	**0.99**	0.74	**1.00**	**1.00**
Integral 14	0.83	0.69	**1.00**	**1.00**
Integral 15	**0.99**	0.73	**1.00**	**1.00**
Integral 16	**0.99**	0.64	**1.00**	**1.00**
Integral 17	**0.99**	0.68	**1.00**	**1.00**
Integral 18	**1.00**	0.86	**1.00**	**1.00**
Integral 19	**0.99**	0.60	**1.00**	**1.00**
Integral 20	**0.99**	0.70	**1.00**	**1.00**
Integral 21	**0.99**	0.74	**1.00**	**1.00**
Integral 22	**0.99**	**1.00**	**1.00**	**1.00**
Integral 23	**0.99**	0.87	**1.00**	**1.00**
Integral 24	0.68	0.72	**1.00**	**1.00**
Integral 25	**0.99**	0.63	**1.00**	**1.00**
Integral 26	**0.99**	0.66	**1.00**	**1.00**
Integral 27	**0.99**	0.98	**1.00**	**1.00**
Integral 28	0.85	**1.00**	**1.00**	**1.00**
Integral 29	0.77	0.76	**1.00**	**1.00**
Integral 30	0.56	0.76	**1.00**	**1.00**
Integral 31	**0.99**	0.56	**1.00**	**1.00**
Integral 32	**0.99**	0.71	**1.00**	**1.00**
Integral 33	0.91	0.70	**1.00**	**1.00**
Integral 34	**0.99**	0.76	**1.00**	**1.00**
Integral 35	0.30	0.44	**1.00**	**1.00**
Integral 36	0.54	0.46	**1.00**	**1.00**
Integral 37 (Dim 1)	0.56	0.49	**1.00**	0.79
Integral 38 (Dim 2)	0.91	0.64	0.29	0.50
Integral 39 (Dim 3)	**0.99**	0.67	0.81	0.50
Integral 40 (Dim 4)	0.70	0.00	0.57	0.00
Integral 41 (Dim 5)	0.60	0.00	0.00	0.00

Table 9. Average power index $\frac{1}{3}\sum_{Z\in\{\mathrm{ARMA,\,Logistic}\}}\hat{Q}^{j}_{i,a\text{ to }b}(Z)$; scores over 0.99 (order 1) or 1.00 (order 2) in bold ($N = 2000$).