*E-mail: dfilipov@dis1.cide.mx. Tel. 727 98 00 Ext. 2708, Fax 727 98 78 (México, D.F.)

CIDE

NÚMERO 136

Dragan Filipovich*

A SOLUTION TO THE PARTNERSHIP PROBLEM UNDER SYMMETRIC TECHNOLOGIES AND DISUTILITIES

Abstract

This paper characterizes the second best solution to the "partnership problem" with budget balancing under the assumption of perfect substitutability of efforts and restrictions on the disutility of efforts. In this latter regard, I assume that all partners have the same disutility index, and that it displays increasing a solute curvature ("risk aversion"). It turns out that, under these conditions, every partner will be called to provide the same level of effort in the second best allocation.

1 Introduction

This paper offers a solution to a special case of the 'partnership problem'. The issue, which goes back to Groves (1973) and Holstrom (1982), is to identify the optimal way of sharing output among partners when their effort levels are not directly observable (and, hence, the sharing rule cannot depend on effort). The problem is to neutralize incentives to free ride. Holmstrom (1982) emphasized that, in general, it will not be possible to achieve the first best allocation under budget balancing, i.e., sharing rules that exhaust the product (a result anticipated in the more general treatment of Groves (1982)).

This result is actually rather intuitive: The first best level of effort requires the marginal increment in output to equal the marginal disutility of effort. This for every partner. Incentive compatibility requires that the marginal share of output equals the marginal disutility of effort. Again, for every partner. But then the only way to make first best efforts incentive compatible is to assign the full marginal increment in output to an individual partner. By budget balance this cannot be done for each and every partner, and the result follows.

Later, Legros and Matthews (1993) characterized the restrictions on the production and utility functions that allow the first best level of effort to be supported under budget balance. Taking my cue from them, I characterize here the second best profile of effort levels that obtains with budget balancing under the assumption of perfect substitutability of efforts and restrictions on the disutility of efforts. I assume that all partner have the same disutility index and that it displays increasing absolute curvature ('risk aversion'). It turns out that, under these conditions, every partner will be called to provide the same level of effort in the second best allocation. This provides a useful benchmark case to evaluate the advantages of alternative arrangements aimed at achieving efficiency. For example, the 'residual claimant' solution pointed out by Holmström (1982) (i.e., the introduction of a partner who claims product but does not contribute effort) which itself is a second best solution (as the claimant cannot participate in production).

2 The Problem

Assume there are $n > 1$ agents denoted by subscripts $i = 1, ..., n$. Agent i chooses his or her effort level e_i from $[0, +\infty)$. This choice is not observable by the other agents. Once every agent has chosen an effort level, a certain publicly observable output y results according to the production function $f(e) : R^n_+ \to R_+$, with $f(0) = 0, f'(.) > 0, f''(.) \leq or \geq 0$. I will specialize and let $f(.)$ take the form $f(\sum_{i=1}^n e_i)$, i.e., agents efforts are taken to be perfect substitutes for each other. Each agent has a utility function of the form $U_i(s_i, e_i) = s_i - v(e_i)$, where $v(0) = 0, v'(.) > 0$ and $v''(.) > 0$. Following the standard convention, I let $e_{-i} = (e_1, ..., e_{i-1}, e_{i+1}, ..., e_n)$. Further, let a sharing rule be a collection of functions $s_i : R_+ \to R_+,\ i = 1, ..., n$. A sharing function s_i gives the share of the collective output allocated by the partnership to its ith member. The partnership problem is then:

$$\max_{\{s_i(.)\}_{i=1}^n, \{e_i\}_{i=1}^n} f\left(\sum_{i=1}^n e_i\right) - \sum_{i=1}^n v(e_i)$$

$$s.t. \quad i)\ \sum_{i=1}^n s_i(y) = y\ ,\ \forall\, y \in R_+,\ with\ s_i(.) \geq 0\ \forall\, i = 1, ..., n$$

$$ii)\ \hat{e_i} \in \arg\max_{e_i}\ s_i\left(f\left(e_i + \sum_{k \neq j} \widehat{e_k}\right)\right) - v(e_i)\ ,\ \forall\, i = 1, ..., n$$

Remarks *1) Note that, given the assumed shape of the partners' utility functions, an allocation is Pareto optimal if and only if it maximizes the objective in the partnership problem. Also note that everyone has the same* $v(.)$ *here.*

2) Constraint i) requires that sharing rules be budget balancing, i.e., *that the sum of the payoffs to the partner fully exhaust the product for any possible realization of output. Moreover, the payoffs to the partners are required to be nonnegative. In other words, punishments are not allowed.*

3) Constraint ii) is an incentive compatibility condition. It requires the partners' prescribed actions to be best responses to the actions of all other partners. This condition captures the non-observability of partners' efforts (together with the requirement that payoffs be conditioned only on output and not on individual efforts). There is an issue of multiplicity of equilibria here, which I will - following the literature- simply ignore.

3 Characterizing Solutions to the Partnership Problem

The solution to the partnership problem, as I have just formulated it, involves choosing a series of functions. The restrictions constraint *ii*) imposes on these functions are not immediately clear. One could take a 'first order' approach (taking the sharing functions to be differentiable), and substitute for constraint *ii*) the first order conditions characterizing agents' best responses. For the procedure to be justified, one would have to make sure that first order conditions are sufficient. This again imposes restrictions on the shape of the sharing functions (this time around, involving second derivatives), and does not really simplify things. Rather than tackling the problem directly in the form just sketched, I take here an indirect approach, following Legros and Matthews(1993): Legros and Matthews characterize the class of partnership problems that allow the first best level of efforts to be supported under budget balance. They do this by specifying conditions directly on the production and utility functions, thus obviating the need to deal with sharing functions explicitly. Following their approach, I proceed to characterize the set of all feasible effort profiles in a way that does not explicitly involve sharing functions. The idea being to identify the (constrained) optimal efforts profile and only then proceed to derive a sharing rule that would support it. Finding the optimal efforts profile is clearly a much more straightforward problem, involving as it does only the choice of n numbers rather than n numbers plus n functions. When one considers that in general there will be many sharing rules supporting a specific effort profile as a solution to the partnership problem, the advantages of this approach become even more evident.

Let E_i be the set of all effort levels that agent i can choose among[1], and let $Y_i(e)$ denote the set of outputs that can be attained by partner i by unilateral deviation from the effort profile e,

$$Y_i(\bar{e}) = \{y \in R_+ | \; f(e_i, \bar{e}_{-i}) = y \;, \; some \;\; e_i \in E_i\}$$

The set of outputs that do not reveal the identity of a non-deviator after a unilateral deviation from e is then

$$Y(\bar{e}) = \cap Y_i(\bar{e})$$

[1] I will switch here to more general notation to emphasize the generality of the argument.

It suffices to concentrate on least-cost deviations,

$$c_i\left(y,\overline{e}\right)=\inf\left\{v_i\left(e_i\right)\mid f\left(e_i,\overline{e}_{-i}\right)=y,\ e_i\in E_i,\ y\in Y_i\left(\overline{e}\right)\right\}$$

If output were shared equally, the most partner i could gain by unilateral deviation from $\overline{e}$ to output y would be

$$g_i\left(y,\overline{e}\right)=\left[y/n-c_i\left(y,\overline{e}\right)\right]-\left[f\left(\overline{e}\right)/n-v_i\left(\overline{e}_i\right)\right]\quad for\ y\in Y_i\left(e\right)$$

With this notation, I can state the following claim:

Proposition 1 *There exists a budget balancing sharing rule sustaining the effort profile* $\overline{e}$ *as a Nash equilibrium in the partnership problem iff*

$$\sum_{i=1}^{n}g_i\left(y,\overline{e}_i\right)\leq 0\ \forall\ y\in Y\left(\overline{e}\right)$$

Proof. The proof of the if part is by construction: Given an effort profile satisfying the condition, I construct a sharing rule that supports it as a Nash equilibrium.

Assume $\overline{e}$ satisfies the condition. Let for all partners i but one, say the jth,

$$s_i\left(f\left(\overline{e}\right)\right)=v_i\left(\overline{e}_i\right)$$

For the jth partner let

$$s_j\left(f\left(\overline{e}\right)\right)=f\left(\overline{e}\right)-\sum_{i\neq j}v_i\left(f\left(\overline{e}\right)\right)$$

Now, for all $y\in Y\left(\overline{e}\right)$ and all partners but the jth, let their payoffs be given by

$$s_i\left(f\left(\overline{e}\right)\right)-s_i\left(y\right)=v_i\left(\overline{e}_i\right)-c_i\left(y,\overline{e}\right)$$

For the jth, let

$$s_j\left(y\right)=y-\sum_{i\neq j}s_i\left(y\right)$$

For $y\notin Y\left(\overline{e}\right)$, for some partner k such that $y\notin Y_j\left(\overline{e}\right)$, let

$$y=s_k\left(y\right)$$

For everyone else,

$$s_i\left(y\right)=0\quad\forall\ i\neq k$$

I claim these rules are budget balancing, feasible under the technology and sustain $\bar{e}$ as a Nash equilibrium. Budget balance is immediate for all output levels. Feasibility for $y \notin Y(\bar{e})$ is obvious. For $y \in Y(\bar{e})$, feasibility is evident from the condition of the proposition, which just states that for $y \in Y(\bar{e})$,

$$\mathsf{f}(\bar{e}) - y \geq \sum_{i=1}^{n} [v_i(\bar{e}_i) - c_i(y, \bar{e})]$$

Finally, to show that these rules sustain $\bar{e}$ as a Nash equilibrium: For $y \notin Y(\bar{e})$, note that, for all agents but the kth, these are just 'trigger' strategies, while the kth agent cannot attain y by unilateral deviation. For $y \in Y(\bar{e})$, the payoffs are constructed in such a way that all agents $i \neq j$ are just indifferent between deviating and not. All what has to be shown is that

$$s_j(\mathsf{f}(\bar{e})) - s_j(y) \geq v_j(\bar{e}) - c_j(y, \bar{e})$$

This follows from the condition of the proposition and the construction of the payoffs.

The only if part follows from the following argument: Assume that for some $y \in Y(\bar{e})$ the condition is not satisfied, i.e.,

$$\mathsf{f}(\bar{e}) - y < \sum_{i=1}^{n} [v_i(\bar{e}_i) - c_i(y, \bar{e})]$$

But then, by budget balance,

$$\sum_{i=1}^{n} [s_i(\mathsf{f}(\bar{e})) - s_i(y)] < \sum_{i=1}^{n} [v_i(\bar{e}_i) - c_i(y, \bar{e})]$$

This implies that for at least one partner j,

$$s_j(\mathsf{f}(\bar{e})) - s_j(y) < v_j(\bar{e}) - c_j(y, \bar{e})$$

and that partner will have an incentive to deviate.

■

This result allows one to rewrite the partnership problem in the following fashion,

$$\max_{e \in \Theta} \mathsf{f}(e) - \textstyle\sum_{i=1}^{n} v_i(e_i)$$

$$where \;\; \Theta = \{e \in E | \textstyle\sum_{i=1}^{n} g_i(y, e) \leq 0 \;\; \forall \, y \in Y(e)\}$$

Let me now cast the special case of the partnership problem I am concerned with here in terms of the formalism just developed. For this purpose I have to refine my notation a little: Let $y(e) \equiv \mathrm{f}(e) = \mathrm{f}(\sum_{i=1}^{n} e_i)$ and $s(e) \equiv \sum_{i=1}^{n} e_i$. Hence, $\mathrm{f}(s(e)) = y(e)$ and $\mathrm{f}^{-1}(y(e)) = s(e)$.

In this special case, one obtains

$$Y_i(\overline{e}) = \{y \in R_+ |\ y \geq \mathrm{f}(s(\overline{e}) - \overline{e}_i),\ \ e_i \in [0, \infty)\}$$

Also, one has

$$Y(\overline{e}) = \cap\{y \in R_+ |\ y \geq \mathrm{f}(s(\overline{e}) - \overline{e}_i),\ \ e_i \in [0, \infty)\}$$

$$= \{y \in R_+ |\ y \geq \mathrm{f}(s(\overline{e}) - \min(\overline{e}_1, ..., \overline{e}_n))\}$$

This follows since if $\overline{e}_j \geq \overline{e}_i$ then $Y_i(\overline{e}) \subset Y_j(\overline{e})$. Finally, one gets

$$c_i(\mathrm{f}(s(\overline{e}_{-i}, e_i)), \overline{e}) = v_i(s(\overline{e}_{-i}, e_i) - [s(\overline{e}) - \overline{e}_i])$$

Using these, the partnership problem can be written

$$\max_{e \in R_+^n}\ \mathrm{f}(\textstyle\sum_{i=1}^{n} e_i) - \sum_{i=1}^{n} v(e_i)$$

$$s.t.\ \ y - \sum_{i=1}^{n} v\left(\mathrm{f}^{-1}(y) - [\textstyle\sum_{j=1}^{n} e_j - e_i]\right) - \mathrm{f}(\sum_{i=1}^{n} e_i) + \sum_{i=1}^{n} v(e_i) \leq 0$$

$$\forall\ \ y \geq \mathrm{f}(\textstyle\sum_{i=1}^{n} e_i - \min(e_1, ..., e_n))$$

$$\forall\ \ i = 1, ..., n$$

Assumption 1 : *Let* $\mathrm{f}(\sum_{i=1}^{n} e_i) - \sum_{i=1}^{n} v(e_i)$ *as a function from* $R_+^n \to R$ *be concave. Assume further* $v'(e) \to 0$ *as* $e \to 0$, $v'(e) \to \infty$ *as* $e \to \infty$, *while* $\mathrm{f}'(0) > 0$ *and* $\mathrm{f}'(s) \to k < \infty$ *as* $s \to \infty$.

Remark ***Having rewritten the problem in this way, it would seem that, even under Assumption 1, the set of feasible effort levels will not necessarily***

be convex[2], *and hence this program will not necessarily be concave . Neither is it immediately obvious to me what kind of conditions one would want to impose to make this program concave.*

Nevertheless, it would still seem possible to say quite a bit about its solution, but not before making an additional assumption on the function v,

Assumption 2 : *$-v$ displays increasing absolute risk aversion.*

Proposition 2 *Under Assumptions 1 and 2, at the (constrained) optimum of the partnership problem, every agent will input the same level of effort, i.e., $e_i^* = \bar{e} > 0 \;\; \forall \, i = 1, ..., n$.*

Proof. The disutility cost of producing any given output level y (i.e., $\sum_{i=1}^{n} v(e_i)$) is minimized at $\frac{f^{-1}(y)}{n}$. This follows from the convexity of v and the assumption of perfect substitutability of efforts. The only real question is here whether, given a feasible effort profile e with $\sum_{i=1}^{n} e_i = m$, and such that not all its components are equal, the effort profile that has every agent inputting $\frac{m}{n}$ is feasible as well. Starting at the constrained optimum level of output y^*, the binding incentive compatibility conditions concern only deviations downwards[3], so it suffices to show that these conditions will continue to be satisfied in going from the original profile to the $\frac{m}{n}$ profile. Let $y = f(s) < y^* = f(m)$, hence $s < m$. For simplicity, I concentrate on the two agent case, though, as far as I can see, the argument generalizes without problems. Incentive compatibility requires

$$f(m) - y \geq \sum_{i=1}^{n} v(e_i) - \sum_{i=1}^{n} v(s - e_i)$$

[2] Basically because $-\sum v(.)$ is concave. One sufficient condition is to take v to be linear. I do not find this interesting since in such a case individual effort levels will remain indeterminate.

[3] For a deviation upwards to be incentive compatible, it must be that

$$y - f(m) \geq \sum_{i=1}^{n} v(s - e_i) - \sum_{i=1}^{n} v(e_i)$$

This means that the surplus is greater at y than at the original level of production, while it is possible to deter deviations downward from y. If it is not possible to deter deviations upward from y, then there is an even better outcome such that deviations downward from it can be deterred. But then the original output level could not have been optimal to start with.

Now, in going from the original (unequal) profile to the homogenous one, the right hand side remains unchanged. Due to the convexity of v, the first sum on the left falls, but so does the second. To establish the desired result it suffices to show that the fall in the first term exceeds that of the second, so that the left hand side overall falls. Define $e_i^s \equiv s - e_i$ and $\alpha \equiv |e_i - \frac{m}{2}|$. Assume $e_1 < e_2$. Note that

$$s - \frac{m}{2} = e_1^s + \alpha = e_2^s - \alpha$$

It follows that

$$\sum_{i=1}^{n} v(e_i) - \sum_{i=1}^{n} v\left(\frac{m}{n}\right) \geq \sum_{i=1}^{n} v(e_i^s) - \sum_{i=1}^{n} v\left(s - \frac{m}{2}\right)$$

(from the convexity of v, increasing absolute risk aversion, and from the fact that $m >\ e_1^s + e_2^s$ and $\frac{m}{2} = e_1 + \alpha = e_2 + \alpha$; see diagram below). ■

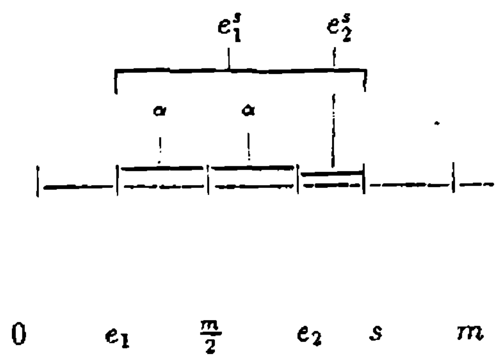

Figure 1.1: Inequality due to Risk Assumption

A simple corollary of this is:

Corollary 3 *Given that, at the optimum, everybody inputs the same level of effort, it is without loss of generality to calculate the optimal effort level using an average product sharing rule, i.e.,* $s_i(y) = \frac{y}{n}$, *for* $i = 1, ..., n$.

After this detour, one comes back to the following very simple form of the partnership problem,

$$\max_{\hat{e}}\ f(n\hat{e}) - nv(\hat{e})$$

$$s.t.\quad \hat{e} \in \arg\max_e \frac{f(ne)}{n} - v(e)$$

Remark *That* $f(ne) - nv(e)$ *is concave follows from Assumption* 1 *above.*

With the problem written in this form, the 'first order' approach becomes a practical option. Note that, under Assumption 1 above, each partner, given the effort levels of the other partners, faces a concave program, and, hence, first order conditions suffice to characterize the partners' best responses. So, this problem can be rewritten yet again as

$$\max_e\ f(ne) - nv(e)$$

$$s.t.\quad \frac{f'(ne)}{n} = v'(e)$$

Remark *I write equality in the condition, rather than inequality, for, at a solution, e will be chosen so that this expression holds with equality. (If* $\frac{f'(e)}{n} < v'(e)$ *then* $f'(e) < nv'(e)$, *and this cannot be the best response, since increasing e is always feasible. Further, at an optimum,* $e > 0$, *since* $v'(0) = 0$ *while* $f'(0) > 0$). *This means that the solution is unique and fully determined by the condition.*

4 Concluding Remarks

As I said in the introduction, this result provides a benchmark to judge the desirability of alternative arrangements aimed at enhancing efficiency. Without knowing what the second -best solution to the partnership problem is, it is hard to say whether wasting the productive potential of one partner (the 'residual claimant') is justified. The same issue arises in other contexts: For example, it would be hard to tell whether inter-firm competition helps to enhance production if one is not in a position to tell how much would have been produced in the absence of such force. Of course, this is only a first step in the task of characterizing the solutions for this problem under more general technologies.

References

[1] J. Farell and E.Lander, Competition Between and Within Teams, *Economic Letters*, **29**, (1989), 205-208.

[2] T. Groves, Incentives in Teams, *Econometrica*, **41,** (1973), 617-631.

[3] B. Holstrom, Moral Hazard in Teams, *Bell Journal of Economics*, **13,** (1982), 324-340.

[4] P. Legros and S. Matthews, Efficient and Nearly Efficient Partnerships, *Review of Economic Studies*, **68,** (1993), 569-611.

[5] P. Legros and H. Matsushima, Efficiency in Partnerships, *Journal of Economic Theory*, **55,** (1991),296-332.