STICKING TO THE WINNERS - HETEROGEOUS FINANCIAL FRICTIONS AND GROWTH

TESINA

QUE PARA OBTENER EL GRADO DE

MAESTRO EN ECONOMÍA

PRESENTA

ERNESTO RIVERA MORA

DIRECTOR DE LA TESINA: DR. DAVID STRAUSS

CIUDAD DE MÉXICO JUNIO, 2016
Agradecimientos.

A mi asesor David Strauss por su dedicación y apoyo incondicional en la realización de este trabajo.
A mis padres por su amor, trabajo y sacrificios en todos estos años. Gracias a ustedes he logrado llegar hasta aquí.
A mi hermano cuyo éxito profesional me motivó a ser mejor cada día.
A mis compañeros por todos los buenos momentos que pasamos juntos.
Contents

1 Introduction .......................................................... 1

2 Model ...................................................................... 7

  2.1 Entrepreneurs ....................................................... 7
  2.2 Workers .............................................................. 9

3 The Credit Market .................................................. 10

4 Individual behavior ............................................... 12

5 Aggregate Equilibrium ............................................. 14

  5.1 Entrepreneurs density .......................................... 14
  5.2 Equilibrium and aggregate variables ...................... 15

6 Heterogeneous and permanent financial constraints .... 19

  6.1 Theory .............................................................. 20
  6.2 Simulations ...................................................... 27
List of Figures

6.1 Graphs of the first simulation. ........................................ 28
6.2 Graphs of the first simulation. ........................................ 30
7.1 Aggregate productivity in terms of persistence of the credit limit $\lambda$ .... 34
Chapter 1

Introduction

A large share of the cross-country disparity in growth rates is been accounted for by differences in government policy (see Rebelo (1990) ). One of the biggest success stories in recent economic history is South Korea which experienced an average growth rate above 8% between 1962 and 1989. At its starting point stood a reform of the financial sector. In the mid-20th century the economy of South Korea consisted almost exclusively of agriculture. When Park Chung Hee became president in 1961 the financial sector was quickly nationalized. The so-called “Chaebol” - a group of family-controlled conglomerates - enjoyed henceforth various preferential treatments among which stands out the access to huge loans from the banking sector.\footnote{The group of chaebol - from chaee “wealth or property” and bol “faction or clan” - include Samsung, LG, Hyundai, and SK Group among others.} By the end of the century, South Korea was one of the export leaders in sectors as electronic devices, motors, telecommunications and others.

In particular, the share of these Korean conglomerates in total credit advanced by financial institutions was over 60\% during the period from 1960-1991 (Lee (2010) ). Besides, the top 30 chaebols’ contribution to the country’s GDP swelled from 9.8\% in 1973 to 29.6\% by
1989 (Choong-Yong (2010)). In 2011, the share of the top 20 chaebol in Korean economy reached an astonishing 85.2% (Murillo y Sung (2013)).

I want to analyze whether the preferential credit agreements given to a small fraction of the economy can have played a decisive role in that development. To that end, I extend the framework of Moll (2014) to allow for heterogeneous financial frictions. Moll analyzed the effect of the financial frictions for total factor productivity in a model with homogeneous borrowing constraints. I find that under the assumption of constant returns to scale, heterogeneous financial frictions imply that the firms that receive better credit conditions will take all available capital in the long-run and the entire economy is characterized by less severe financial frictions. The development in Korea seems to depict that theoretical long-run phenomenon.

More specifically, I show that without government intervention the unique equilibrium displays homogeneous budget constraint. Then, I argue how government intervention can cause looser borrowing constraints for some firms while tighter ones for other firms by monitoring more closely the first group of firms. The resulting heterogeneous financial frictions dampen the negative effect of financial constraints on capital misallocation and total factor productivity. Besides, I show that the more persistent the heterogeneous borrowing constraints are the less severe are the consequences of financial frictions. Both results help to motivate the policy of the Korean government in the 60’s. Importantly, these results do not depend on the proportion of firms that benefit from the government intervention nor on the cost for the rest of the economy in terms of worse borrowing conditions - at least in the long-run.

---

The main mechanism of the policy is that the government can find a way to add heterogeneity to the financial market by increasing the credit limit for a fixed group of selected firms and decreasing it for the rest. The reason is that if the government monitors closely some selected firms (at the expense of less monitoring of the rest of the economy) competitive banks are willing to lend more to this selected group of firms and thereby increase their borrowing constraints.\(^3\) In contrast, the rest of the economy will face impoverished borrowing constraints as they are less monitored. This government intervention implies a short-run negative effect if the average borrowing limit for entrepreneurs decreases and, consequently, total factor productivity decreases. However, the policy implies that the share of capital employed by the "chaebol" (that receive preferential credit conditions) increases over time. This is due to the linearity of the savings function of the firm in the credit limit, i.e. a higher credit limit implies a larger savings rate. Therefore, the "chaebol" will experience a strictly greater rate of growth. Consequently, if the policy prevails and the government sticks to their "chaebol", they theoretically will take all the capital available in the economy in the long-run. In the limit, the economy behaves as if all firms faced the better credit conditions and, therefore, will experience higher growth rates and total factor productivity. This extreme case occurs only if the selected group of firms is fixed over time. I extend this result to show that a higher persistence in the preferentially treated firms always leads to higher growth and productivity.

A caveat worth mentioning is that the constant returns to scale assumption is essential to find a linear savings function and therefore easily find the aggregate variables of the economy. However this assumption may lead to a biased theoretical result of the "chaebol" policy. This is due to the fact that the policy implies a decrease in the number of active firms and, by definition, an assumption of decreasing returns to scale would decrease the total production. Nevertheless, I state that for small degrees of decreasing returns to scale, the positive effect of the policy should prevail as I found in the data of South Korea.

\(^3\)See Banerjee y Newman (1993) and section 3 below.
There exist a large theoretical literature about financial frictions in growth development, for early contributions see Banerjee y Newman (1993) and Galor y Zeira (1993). Similar to my model, in Galor y Moav (2004), in early stage of development capital should go to the agents with a higher propensity to save. Also, Piketty (1997) introduced credit rationing into a Solow model. 4

It is worth to mention certain theoretical papers that illustrates the importance of financially imperfect markets, for example Aghion y Bolton (1997) developed a model where they analyze the trickle-down effect of capital accumulation and whether there are persistent income inequalities. F. J. Buera, Kaboski, y Shin (2009) show how financial frictions are able to explain a large part of country differences in the total factor productivity in a model of two sectors.

There also exists a large empirical literature about the importance of financial frictions in economic development, for example in Beck y Demirguc-Kunt (2006). They show that small and medium size firms have proportionally less access to credit than larger firms and therefore in Beck, Demirguc-Kunt, Laeven, y Levine (2008) it is shown that the financial market development generates a disproportionately positive effect on the small size firms. There exist also a large literature about government intervention in development and growth. For example, Itskhoki y Moll (2014) introduce a tax policy that is pro-business in early stages and “pro-labor” in final stages that as a result accelerates the transition of capital accumulation of the country. 5

---

4For more literature about misallocation see also Hsieh y Klenow (2007), Midrigan y Xu (2010) and Banerjee y Dufló (2005).
5See also Restuccia y Rogerson (2008) and F. J. Buera, Moll, y Shin (2013) for more literature on development policies.
My work is closely related to two papers in the financial frictions literature. First, it is related to Moll (2014), as I use his setup on financial frictions as a starting point to analyse the effect of heterogeneity in the borrowing constraints of the entrepreneurs. I extended the work of Moll in order to easily track the consequences of adding heterogeneous borrowing constraints. Therefore I use both crucial assumptions that the production function exhibits constant returns to scale and entrepreneurs live infinitely. Agents possess heterogeneous talent governed by continuous Markov process that will lead the talent to abruptly orbit around a certain value. However, in this thesis, I will analyse the case when entrepreneurs can face different credit limits. The possible different values of the credit limit $\lambda$ are in a continuum or discrete set and can be fixed or changed over time. The second paper closely related to my work is Song, Storesletten, y Zilibotti (2011). This paper also displays heterogeneity in the borrowing constraint of firms, in particular they assume that state-owned firms in China face no borrowing constraints but are less productive than private firms. The state-owned firms are able to survive in early stages of development because of their better access to credit. However, in contrast to my model, in Song et al. (2011) the firms with better borrowing conditions will perish in the long run due the higher productivity of private firms. Therefore, in survival terms, the talent dominates the access to credit. In my model, if talent and access to credit are independent the access to credit determines survival in the long-run.

The rest of the paper is organized as follows. In section 2, I lay out the model economy. Then, I discuss both the laissez-faire equilibrium on the credit market as well as the consequences of a potential government policy in (3). Chapter 4 characterizes individual decisions given prices and in (5) I define the equilibrium and calculate the aggregate equilibrium variables in the economy. In chapter 6, I analyze how a fixed heterogeneous system of borrowing constraints leads to a positive effect on total factor productivity in the long run with respect...
to homogeneous financial constraints. I discuss the case when the credit limit of each entrepreneur is governed by a continuous Markov process in (7) and conclude in chapter 8.
Chapter 2

Model

Firstly, it is discuss the agents in this economy, as in Moll (2014) the economy will consist of two types of agents, entrepreneurs and the labor force. For simplicity purposes I will suppose that both mass of agents will be constant along the time and there will not be any transition between the types.

2.1 Entrepreneurs

There is a continuum of entrepreneurs that differ potentially in three characteristics, productivity or talent ”z”, credit limit ”λ”, and (endogenous) wealth ”a”. Both z and λ can face shocks over time and, using the law of large numbers, the path of the characteristics will be deterministic. I suppose that the mass of the entrepreneurs w.l.o.g. is equal to 1 and characterized by a density distribution function $g_t(a, z, \lambda)$ at time $t$. Their preferences are represented by the utility function

$$E_0 \int_0^\infty e^{-\rho t} \log(c_t) \, dt,$$
where \( \rho \) is the discount rate. Each entrepreneur possesses his own firm with a production function given by

\[
y = f(z, k, l) = (zk)^{\alpha}l^{1-\alpha},
\]

where \( k \) and \( l \) are the units of capital and labor they possess and \( \alpha \in (0, 1) \) is fixed and equal for all the entrepreneurs. Capital depreciates at rate \( \delta \).

There are both a competitive labor and capital market with wage \( w_t \) and rental rate \( R_t \). The rental rate has to satisfy \( R_t = r_t + \delta \), where \( r_t \) is the interest rate. The wealth accumulation of each entrepreneur satisfies

\[
\dot{a} = f(z, k, l) - w l - (r + \delta) k + r a - c.
\]

As mentioned, credit markets are not perfect and thus each agent faces the collateral constraint

\[
k \leq \lambda a,
\]

(2.1)

i.e. the credit that each agent can request is limited by his own wealth and the credit constraint \( \lambda \) which potentially differs among agents. Then the value of \( \lambda \) represents the credit market rigidity for each agent, the higher value of \( \lambda \) for the entrepreneurs, the lower the credit market friction. I will suppose that the entrepreneur talent changes in time for each entrepreneur, in fact, I will model each individual talent as an independent stochastic process, this is

\[
dz = \mu(z)dt + \sigma(z)dW
\]

where \( \mu(z) \) is called the drift term and \( \sigma(z) \) the diffusion term. I will also suppose that the process is ergodic and admits an stationary distribution.
In chapter 6 I will suppose that the credit limit $\lambda$ is fixed for each entrepreneur, but in chapter 7 I will suppose that the dynamic of $\lambda$ follows an independent stochastic process similar as $z$’s.

2.2 Workers

For math simplicity, I will suppose that the labor force is conformed by a fixed mass $L$ of workers. They will have the same preferences as the entrepreneurs, but they will not have access to any savings technology. I will also suppose that each worker supplies one unit of labor inelastically. since they are hand-to-mouth agents and do not have any decision to make. I will omit them from the remainder of the analysis.
Chapter 3

The Credit Market

Before continuing with the analysis it is essential to lay out a model of the credit market. I follow first the story of Banerjee (1993) and then modify it. I do abstract from bankruptcy issues, lenders can always fully repay. Now, suppose that borrowers put their whole wealth $a$ as collateral in order to borrow $k$. He may now avoid his obligation to repay and flee. In that case, he looses his collateral worth $ra$. Fleeing does not affect economic outcome $\pi(k)$. However, at the end of the period, there is a small probability $p$ that he gets caught. In that case, he will receive a non-monetary punishment $F$. I could assume that $F$ depends linearly of the capital borrowed and therefore also of the collateral. Then I can assume that $F(a) = ca$ for a certain constant $c$. Hence, any lender strictly prefers to flee whenever

$$\pi(k) - p F(a) > \pi(k) - (1 + r) k + (1 + r) a \iff k > a + \frac{p F(a)}{r}.$$ 

Thus, as financial institutions anticipate lenders’ behavior they restrict the borrowing up to

$$k_{max} = a + \frac{p F(a)}{1 + r} = a \left(1 + \frac{p c}{1 + r}\right)$$

(3.1)
Which is consistent with the financial restriction given in (2.1).

Under the assumption of a same caught probability of \( p \) and punishment \( F(a) \) for all individuals it is straightforward to observe from the last equation that this is the unique laissez-faire equilibrium. Let me modify now slightly the model. More specifically, suppose that the government possesses a certain budget constraint \( B \) which determines the monitoring possibility which in turn determines the probability to get caught, \( p \). Thus, the government can choose to monitor certain firms more closely (the chaebol to which it extended strong ties) at the expense of monitoring other firms. In other words, it can increase \( p \) for some firms while decreasing \( p \) it for the rest of the economy.

Inspection of equation (3.1) reveals that such a policy increases the borrowing limit for some firms and decreases it for the rest of the economy leading heterogeneity on the credit market. However, I would like to analyse the effect of this policy in the average credit limit. Suppose firstly that every entrepreneur has the same credit limit \( \bar{\lambda} \) and I impose the described policy. As mentioned, this would lead the value \( \lambda \) to be heterogeneous between the entrepreneurs.

Suppose that the probability of getting caught \( p \) is a non decreasing function of the budget constraint \( B \). Then, it is clear by the definition of concavity and the equation (3.1) that the more concave the function \( p(B) \) is, the greater the difference of \( E(\lambda) \) and \( \bar{\lambda} \). Since it is not hard to believe certain concavity in the function of getting caught \( p(B) \), I could expect that the described policy could have a negative impact in the average credit constraint.
Chapter 4

Individual behavior

In this section, I analyze the decisions entrepreneurs will take to maximize their utility function given the prices of inputs and their financial constraints. \(^1\) This first lemma shows us the demands of inputs of each entrepreneur given the prices. It is straightforward from the constant returns to scale and the financial restriction that the inputs and the profits will be a linear function of the actual wealth \(a\) multiplied by the credit limit \(\lambda\). In the aggregate section, the linearity of these variables will allow me to easily find the aggregate variables of the economy.

**Lemma 4.0.1.** *Given the market prices \(r, w\) the optimal demand of inputs and profits of each entrepreneur will be*

\(^1\)This section draws heavily from Moll (2014).
and the productivity cutoff is defined by $z\pi = r + \delta$

This lemma states that a firm will only produce if it is above a certain productivity level. In that case, the firm will borrow all the available credit that it disposes and then demand the labor that maximizes its profits according to the prices. Using the next lemma, I will be able to characterise the evolution of the wealth $a$ of each entrepreneur as linear function that depends of the talent $z$ and the financial constraint $\lambda$.

**Lemma 4.0.2.** Given the market prices $r, w$ the optimal savings policy function will be

$$\dot{a} = s(z, \lambda)a, \quad \text{where } s(z, \lambda) = \lambda\max\{z\pi - r - \delta, 0\} + r - \rho$$

Note that the savings function $s$ is non decreasing in $z$ and $\lambda$. I will see that in general, the interest rate $r$ will be inferior than the discount factor $\rho$. This inequality will lead to a decrease in the individual entrepreneur wealth for all the non producing firms as expected, and a high rate of growth for most of the producing firms. Now, using these two lemmas, I am ready to find the aggregate variables of the economy.
Chapter 5

Aggregate Equilibrium

In this section I will find the aggregate variables of the economy like the total factor productivity in period $t$ and, given the density distribution function $g_t$ of the three heterogeneous variables, the wealth, the talent and the access to credit. Knowing this joint distribution, I could compute the prices, the total (or individual) demand of inputs of the entire economy and the profits.

5.1 Entrepreneurs density

In order to get the aggregate variables I need to know the distribution across time of three characteristics that entrepreneurs have, their talent $z$, their credit limit $\lambda$ and their wealth $a$. As I said earlier I will suppose w.l.o.g. that there is a continuum of entrepreneurs with mass equal to one.

Let’s denote $g_t(a, z, \lambda)$ the density function that represents the distribution of these three
characteristics. It should be a positive function and, certainly, it should integrate to one for each $t$, i.e.,

$$\int \int g_t(a, z, \lambda) \, da \, dz \, d\lambda = 1$$

Now, as I have constant returns to scale, I just need the total amount of capital for each pair $(\lambda, z)$ to compute the aggregate variables and dynamics of the economy. Suppose that the aggregate capital is denoted by $K_t$. Then, in order to save notation, it will be convenient to define the wealth held by the entrepreneur’s type $(z, \lambda)$ and the marginal density function of the talent $z$ as

$$\omega_t(z, \lambda) := \frac{1}{K_t} \int_0^\infty g_t(a, z, \lambda) \, da \quad \text{and} \quad \omega_{z,t}(z) := \int_1^\infty \omega_t(z, \lambda) \, d\lambda$$

I also define the cumulative distribution function as

$$\Omega_t(z) := \int_0^z \omega_{z,t}(x) \, dx$$

i.e., $\Omega_t(z)$ is the total amount of wealth of all the entrepreneurs with talent less or equal than $z$.

As I will see later, knowing the total amount of capital $K_t$ and the wealth density $\omega_t$ would give enough information to calculate all the aggregate variables of the economy and also the dynamics of $\omega_t$ and therefore, the dynamics of all the aggregate variables.

### 5.2 Equilibrium and aggregate variables

Before finding the aggregate variables, I shall state the definition of a competitive equilibrium of the economy. It will be defined in the same way as Mooll (2014):
Definition 5.2.1. I define a competitive equilibrium as a path of prices $r(t), w(t)$ such that:

- All the entrepreneurs (and workers) maximize their lifetime expected utility subject to their budget constraint, taking the prices as given.
- The capital and labor markets clear, this is

$$K_t := \int \int \int a g_t(a, z, \lambda) \, da \, dz \, d\lambda = \int \int \int k_t(a, z, \lambda) \, g_t(a, z, \lambda) \, da \, dz \, d\lambda$$

$$L = \int \int \int l_t(a, z, \lambda) \, g_t(a, z, \lambda) \, da \, dz \, d\lambda$$

Where the left hand side and the right hand side are the total supply and total demand of inputs respectively.

Now, using the optimal demands of inputs given in Lemma 1, I can substitute $k_t(a, z, \lambda)$ in the market-clearing condition for $K$ to obtain

$$\int_{\tilde{z}}^{\infty} \int_{1}^{\infty} \int_{0}^{\infty} \lambda a g_t(a, z, \lambda) \, da \, d\lambda \, dz = K_t$$

and therefore rearranging

$$\int_{\tilde{z}}^{\infty} \int_{1}^{\infty} \lambda \omega_t(z, \lambda) \, d\lambda \, dz = 1$$

Then I can observe that $\tilde{z}$ is a function of the quality of the credit markets. In particular, when $\lambda$ is constant among all the entrepreneurs, It is easy to see that

$$\lambda(1 - \Omega_t(\tilde{z})) = 1$$
This equation directly tells us a direct and positive relation between the limit credit and the cut-off productivity level $\tilde{z}$. Then I can conclude that an economy with high frictions in the financial market will have a less average productivity. Now, to obtain the aggregate dynamics of the capital stock, integrate $\dot{a}$ over all the entrepreneurs to obtain

$$
\dot{K}_t = \int_0^\infty \int_1^\infty \int_0^\infty \dot{a}(a, z, \lambda) \, da \, d\lambda \, dz
$$

\[ = K_t \int_0^\infty \int_1^\infty s_t(z, \lambda) \, \omega_t(z, \lambda) \, d\lambda \, dz \]

\[ = K_t \int_0^\infty \int_1^\infty [\lambda \max\{z\pi(t) - r(t) - \delta, 0\} + r(t) - \rho] \, \omega_t(z, \lambda) \, d\lambda \, dz \]

As I can see, the aggregate capital accumulation is also a linear function of the current level of capital. This is directly a result of the linearity of the individual entrepreneur savings function. Now it is needed to find the productivity cut-off and the total factor productivity of the economy in terms of the distribution of the wealth among the entrepreneur types $(z, \lambda)$

**Proposition 5.2.2.** Given a time path for wealth shares $\omega_t(z, \lambda), t \geq 0$, aggregate quantities satisfy

$$
Y = ZK^\alpha L^{1-\alpha}
$$

$$
\dot{K} = \alpha ZK^\alpha L^{1-\alpha} - (\rho + \delta)K
$$

where

$$
Z = X^\alpha := \left( \int_\vec{z}^\infty \int_1^\infty \lambda z \, \omega(z, \lambda) \, d\lambda \, dz \right)^\alpha
$$

and the productivity cutoff $\vec{z}$ is defined by

$$
\int_\vec{z}^\infty \int_1^\infty \lambda \, \omega_t(z, \lambda) \, d\lambda \, dz = 1
$$
and the prices are given by

\[ w = (1 - \alpha)ZK^\alpha L^{-\alpha} \quad r = z^\alpha Z^{\frac{\alpha - 1}{\alpha}} K^{\alpha - 1} L^{1-\alpha} - \delta \]

All the proofs are relegated to the appendix.

Now let’s interpret the wealth distribution \( \omega \) among the entrepreneur types \( (z, \lambda) \) as a probabilistic joint density of the random variables \( \lambda \) and \( z \). Then the total factor productivity \( Z \) could be understood as a function of an expectation:

\[ Z = E_\omega[\lambda z 1(z \geq \bar{z})]^\alpha \]

Therefore, I can argue that ceteris paribus, the higher the limit credit \( \lambda \) the higher this expectation will be, leading to an increase in the total factor productivity. I shall note that an increase in the parameter \( \lambda \) could affect the cutoff value \( \bar{z} \) because as higher is the limit credit, more capital the talented entrepreneurs could demand and, as I have a limited supply, the prices will update until the capital market clears. This leads to a fewer firms producing and a higher cut-off limit \( \bar{z} \). Nevertheless, it is not hard to see that the capital is shifted to more talented entrepreneurs and therefore the total factor productivity would be increased.
Chapter 6

Heterogeneous and permanent financial constraints

In this section I will discuss the effect that a permanent and heterogeneous friction in the financial system could lead. It is known from Moll (2014) that under an homogeneous financial friction system, the higher the limit of financial constraint a economy has, the less number of firms will be producing and the higher the total factor productivity of the economy will be. Now, suppose that there are two groups of entrepreneurs, the $H$ and the $L$ group. Both groups will have a different and fixed level of credit limit and the $H$ group will have a higher access to credit, this means $\lambda_H > \lambda_L$. Then I will see that under constant returns to scale the $H$ group will be holding all the capital in the economy in the long run. Therefore, the economy will behave like it has only entrepreneurs with the higher credit limit $\lambda_H$ leading to a higher total factor productivity. It is important to note that I am assuming that the different values of $\lambda$ have not correlation with the talent path of individuals $z(t)$. 
6.1 Theory

In this subsection I will theoretically demonstrate, under certain assumptions, the positive effects that heterogeneity in the credit limit $\lambda$ could lead in the short and long run. Firstly let’s remind the proposition given by Moll (2014) where he describes the dynamics of the wealth density.

**Proposition 6.1.1.** Let a wealth density $\omega_t$ to be defined by

$$
\omega_t(z, \lambda) = \frac{1}{K_t} \int_0^\infty a g_t(a, z, \lambda) \, d\lambda
$$

with the dynamics given by $\dot{\omega} = s_t(z, \lambda) \omega_t$. Then the wealth shares $\omega_t(z, \lambda)$ obey the second-order partial differential equation

$$
\dot{\omega}_t(z, \lambda) = \left[ s_t(z, \lambda) - \frac{K}{K_t} \right] \omega_t(z, \lambda) - \frac{\partial}{\partial z} \left[ \mu(z) \omega_t(z, \lambda) \right] + \frac{1}{2} \frac{\partial^2}{\partial z^2} \left[ \sigma^2(z) \omega_t(z, \lambda) \right]
$$

The last equation is directly derived by the Kolmogorov forward equation which determines the dynamics of a probability distribution that is changing over time by a deterministic movement and by an aleatory factor, in this case determined by the shocks over time of the variable $z$. It is worth to mention that the last proposition is also true for subgroups of entrepreneurs instead of taking all the entrepreneurs in the economy. For example taking the group $H$ of entrepreneurs the wealth density would be defined as

$$
\omega_{H,t}(z, \lambda) = \frac{1}{K_{H,t}} \int_0^\infty a g_{t}(a, z, \lambda) \, d\lambda
$$
and then
\[
\dot{\omega}_{t,H}(z, \lambda) = \left[ s_t(z, \lambda) - \frac{\dot{K}_{H,t}}{K_{H,t}} \right] \omega_{t,H}(z, \lambda) - \frac{\partial}{\partial z} \left[ \mu(z) \omega_{t,H}(z, \lambda) \right] + \frac{1}{2} \frac{\partial^2}{\partial z^2} \left[ \sigma^2(z) \omega_t(z, \lambda) \right]
\]

Now it is known now that initially the credit limit should be equal across every entrepreneur. In order to continue, let’s introduce a way to define heterogeneity in the credit limit derived by some exogenous intervention. Let’s define a mean preserving shock over a unique value of the credit limit.

**Short run effects**

Now that I have characterized the dynamics of the wealth distribution, I am ready to observe the effects that an heterogeneous financial system could lead. I will begin with the short run effects. I will show that a shock that generates heterogeneity without changing the mean of the credit limit \( \lambda \) produces an immediate positive effect in the total factor productivity of the economy. Firstly let’s define the formal definition of this type of shocks.

**Definition 6.1.2.** Suppose that firstly there is a unique limit of credit given by \( \bar{\lambda} \). Now I will define a mean preserving spread at time \( t \) as new permanent and independent distribution of credit limit \( \lambda \) for the individuals with two possible values, \( \lambda_H \) and \( \lambda_L \) such that

\[
E(\lambda) = \bar{\lambda}
\]

I will denote \( H \) and \( L \) the entrepreneurs group with credit limit \( \lambda_H \) and \( \lambda_L \) respectively.

With this definition I am ready to demonstrate the positive effects that the new allocation
of financial constraint cause instantly. The intuition is that the new set of constraints start to assign more resources to the less constrained firms.

**Lemma 6.1.3.** Suppose that the economy has only one credit limit denoted by $\bar{\lambda}$ and is in its steady state. Then at time $t^*$ a mean preserving spread on $\lambda$ occurs. Then the productivity $Z$ and the cutoff $\bar{z}$ are continuous in $t^*$ and

$$\dot{z}(t^*) > 0 \quad and \quad \dot{Z}(t^*) > 0$$

i.e fewer firms will be producing and the total factor productivity will start growing.

The intuition of this result is that the path of $Z$ across time depends entirely on the path of $\bar{z}$. The less the value of $z$ the fewer firms will be producing and the greater productivity average that the producing firms will have. Then if $\bar{z}$ is continuous and increasing, $Z$ will be also continuous and increasing.

Showing that $\bar{z}$ is continuous is very easy since the spread of $\lambda$ is independent of the talent and, as I have a mean preserving spread, the aggregate demand of capital is the same for each level of talent. It can be also found that if the shock does not preserve the mean of the credit limit $\lambda$, then there would be a discontinuity in both $Z$ and $\bar{z}$. Now, to show that the cutoff (and the productivity) will be increasing in the short run, I can see using some simple math that the derivative of $\bar{z}$ depends on the values of $E[\lambda s(z, \lambda) | z]$ for all the values of $z$ greater than $\bar{z}$. Now, before the shock this value is equal to $\bar{\lambda} s(z, \bar{\lambda})$, but after the shock the expectation become greater since $\lambda$ is positively correlated with $s(z, \lambda)$ and the bigger the variance the bigger the change in the derivative would be. Therefore, the more disperse the shock is, the bigger the positive effect that the growth would have.
Long run effects

I want to know why this positive effect is happening and if it will be still happening in the long run. As I said earlier, the intuition tells us that there should be a positive effect in the long run due the change in the relative wealth of each group of entrepreneurs. The higher the credit limit, the higher the total savings a group will have and, therefore, in the long run the entrepreneurs with the highest credit limit $\lambda$ will be the only firms producing. I state earlier that the short run positive effects of heterogeneity depend on the mean of the value of $\lambda$. If the mean of $\lambda$ decreases, the short run effects are ambiguous. Nevertheless this kind of shocks still produces positive long run effects no matter the new mean value of $\lambda$. Let’s continue introducing this kind of shocks.

Definition 6.1.4. Suppose that firstly there is a unique limit of credit given by $\bar{\lambda}$ Now I will define a spread shock at time $t$ as a new permanent and independent distribution of credit limit $\lambda$ for the entrepreneurs with two possible values $\lambda_H$ and $\lambda_L$ with $\lambda_H > \bar{\lambda} > \lambda_L$. Let’s also define $\omega_{H,t}$ and $\omega_{L,t}$ as the conditional distributions of $\omega_t$ subject to $\lambda = \lambda_H$ and $\lambda = \lambda_L$ respectively. This is

$$\omega_{H,t}(z) = \frac{1}{K_{H,t}} \int_0^\infty ag_t(a, z\lambda_H) \, da$$

and

$$\omega_{L,t}(z) = \frac{1}{K_{L,t}} \int_0^\infty ag_t(a, z\lambda_L) \, da.$$

Here, $K_{H,t}$ and $K_{L,t}$ are the total wealth of the groups with credit limit $\lambda_H$ and $\lambda_L$ respectively.

Note that in this case I am not assuming that the distribution preserves the mean, I do
not need this assumption for the long run results. It is also worth saying that the definition of the shock could be generalized to a distribution that takes more than two values and even a continuous distribution without changing the results. The assumption of just two values has been taken just for simplicity. Let’s introduce a lemma that show us a necessary and sufficient condition of the increase in the relative capital of a group of entrepreneurs. This lemma is not needed for the main long run results but gives us a nice intuition about why the economy starts to behave like it has under the higher credit limit.

Lemma 6.1.5. Suppose that a simple spread shock on \( \lambda \) has occurred. Then

\[
\left( \frac{K_H}{K_L} \right) > 0 \quad \Leftrightarrow \quad \mathbb{E}_\omega[s(\lambda, z) \mid \lambda = \lambda_H] > \mathbb{E}_\omega[s(\lambda, z) \mid \lambda = \lambda_L]
\]

Where \( \mathbb{E}_\omega \) is the conditional expectation with respect to the density distribution function \( \omega_t \).

If I interpret the wealth density as a probability density, then the relative savings growth of one group of entrepreneurs depends only on the relative expectation of savings. Note also that this result does not depend on the proportion of individuals of each group. Therefore, even if a very small group of entrepreneurs have a higher savings function they could accumulate a very large part of the total capital stock of the economy over time. Even more, if

\[
\mathbb{E}_\omega[s(\lambda, z) \mid \lambda = \lambda_H] > \mathbb{E}_\omega[s(\lambda, z) \mid \lambda = \lambda_L]
\]

(6.1)
holds permanently over time, then the group with a higher credit limit will accumulate all the capital in the economy.
Unfortunately, showing equation (6.1) for all $t$ in the future is not so easy because the densities of both groups are changing over time differently. However I can show that at least in the long run steady-state the growth rate of the high-limit entrepreneur group should be higher.

**Lemma 6.1.6.** if $\lambda_L < \lambda_H$, then in the long-run steady state of the economy the growth rates of capital of both groups satisfy

$$\frac{\dot{K}_H}{K_H} > \frac{\dot{K}_L}{K_L}$$

With the above result I can easily see that in the long run the high credit limit group of entrepreneurs will be accumulating all the capital in the economy, and finally only the less constrained group of firms will be producing.

Now I will introduce the main result that claims that in the long run the capital stock of the low-credit limit group could be ignored, and therefore the aggregate variables like the total factor productivity will be approximately the same as an economy with just the high credit limit constraint. i.e. , the economy will behave like it only had the credit limit value $\lambda_H$ for all the entrepreneurs. To demonstrate this, I firstly I show that the capital $K_L$ tends to zero, and then the equation that govern the dynamics of the rest of the capital $K_H$ solves approximately the same equation of an economy that only has the credit limit $\lambda_H$.

**Theorem 6.1.7.** Suppose that a steady-state economy with only one credit limit has a simple spread shock with two new limits $\lambda_L < \lambda_H$ at time $t^*$. Then, independently of the proportion of the high credit limit group, in the long run

$$\lim_{t \to \infty} \frac{K_{H,t}}{K} = \lim_{t \to \infty} \frac{K_{H,t}}{K_{L,t} + K_{H,t}} = 1,$$
the wealth dynamics are given approximately by

\[ \dot{\omega}(z, \lambda_i) \, dz \approx \left[ s(z, \lambda_i) - \frac{\dot{K}_H}{K_H} \right] \omega(z, \lambda_i)dz - \frac{\partial}{\partial z} \mu(z)\omega(z, \lambda_i)dz + \frac{1}{2} \frac{\partial^2}{\partial z^2} \sigma^2(z)\omega(z, \lambda_i)dz \]

\( r, w, \pi, z \) and \( Z \) will be approximately equal to those in an economy where the entrepreneurs of credit \( \lambda_L \) do not exist. Therefore, the total factor productivity behaves like it had just the highest credit limit \( \lambda_H \) and the entrepreneurs with the low limit credit will not produce. This is the main result of the entire thesis. Imagine that firstly the economy faces an homogeneous credit limit \( \bar{\lambda} \) and then the government decide to introduce a policy described in chapter three that induces a higher credit limit \( \lambda_H \) for a fixed group of firms with the cost of inducing a lower credit limit for the rest of the firms. This result claims that this policy would have a positive effect in the long run if this government intervention sticks infinitely over time to the selected group of firms. In the case of transition of groups among the preferred group of firms then the result does not hold with the same efficiency but still generates a small positive effect as I will see in the the next section.

This result could be easily generalized to a more general spread shock. The heterogeneous values of the credit limit could take a discreet of a continuous set of values. The result would be analogous to the previous case. Call \( \lambda_{\text{max}} \) the maximum credit limit that the entrepreneurs face. Then the economy will asymptotically behave as if all the entrepreneurs face the same credit limit \( \lambda_{\text{max}} \).

It should be emphasized that this result depends entirely on the assumption of constant returns to scale. However, there is empirical evidence of non linearity in the production function. For example, with a production function with decreasing returns to scale the final result should be a opposite sum of two effects. The pure positive effect of the allocation of
capital to better firms, and the negative effect of the lower number of firms producing with the decreasing returns to scale assumption.

6.2 Simulations

In this subsection I will analyze if the results obtained in the theory subsection agree with a program that simulates the behavior of the density of entrepreneurs. Here I will assume that the process of $z$ follow a Ornstein-Uhlenbeck process. As Moll (2008) did, this assumption will allow to capture the effect of persistence in the heterogeneity of $z$

$$d \log z = \frac{-1}{\theta_z} \log z dt + \sigma_z \sqrt{1/\theta_z} dW$$

The reason why I use this process is because it captures the effect of persistence in the evolution of the talent with the parameter $\theta_\lambda$ and the variance of the process given by $\sigma_z$.

In the simulations, I will declare a grid of point that will represent the wealth density given by $\omega$. for each period of time I will firstly calculate the cutoff $z$ of the entrepreneurs, the prices $r, w$, the aggregate variables and the dynamics of $\omega$. Using the dynamics I calculate the near path of $\omega$ and recalculate all the remaining variables again and so on. All the simulations were made with the standard parameters $\rho = 0.05, \delta = 0.05, \alpha = 0.33$, where $\rho$ is the future discount factor $\delta$ is the depreciation rate, $\alpha$ the capital intensive parameter of the production function.
First simulation

The goal of this simulation is to reproduce the theoretical results of a mean preserving shock in the credit limit $\lambda$. I will simulate 3 different economies. The first, represented by the slashed line, will have a constant credit limit of $\lambda_1 = 4$. The second, the dotted line, will have a constant credit limit of $\lambda_2 = 2$. The third, represented by a solid line, will have initially a credit limit of $\bar{\lambda} = 3$ but then on the period $t^*$, represented by a black line, there will be a mean preserving spread shock leading this economy to a two new credit limits, $\lambda_L = 2$ and $\lambda_H = 4$ each value will be assigned randomly to each half of the entrepreneurs population. All the other parameters will be the same in the three economies.

What is represented in the graphs are the value of $z(t)$ and $Z(t)$ over time.

Figure 6.1: Graphs of the first simulation.

(a) $z$ cutoff over time
It can easily be seen that both values are continuous at the period of the shock and that the economy with heterogeneity slowly converges to the economy with the highest credit limit. Therefore I conclude that a mean preserving spread shock not only affects the short run productivity but also in the long run the productivity asymptotically behaves as an economy with only the high credit limit $\lambda_H$.

**Second simulation**

In this simulation I will again simulate three different economies. The first, will be represented by the slashed line and will have a high financial constraint of $\lambda_1 = 4$. The second, the dotted line, will have a low credit limit of $\lambda_2 = 2$. And the last economy, represented by a solid line will have initially a credit limit of the mean of the first two economies given by $\bar{\lambda} = 3$. Then at certain time $t^*$, represented by a black line, the third economy will have a non-mean preserving spread in their financial limits with 95% of the entrepreneurs having a low limit of $\lambda_L = 2$ and the rest 5% will have a high limit $\lambda_H = 4$. 
In contrast with the first simulation, there is not a mean preserving spread of the credit limit $\lambda$. Instead the mean of $\lambda$ decreases to 2.10 at the time of the shock. In this case I allow a high credit limit just for a very small percent of the entrepreneurs and a low credit limit for the rest. As a result, the aggregate productivity decay immediately after the shock, meaning a negative effect in the short run. However, as I can see in the graphs the productivity will slowly start to grow because of the capital accumulation of the small percentage of preferred firms. Over time, the $H$ group of entrepreneurs will be hoarding all the available capital. Finally in the long run the productivity would converge to the the productivity of the economy that has an homogeneous credit limit $\lambda_H$. This means a positive effect on productivity in the long run.
Chapter 7

Heterogeneous and transitory financial constraints

In this section I will introduce a different heterogeneous financial system. In this generalization I will define the financial constraint as a continuous time version of a Markov process similar to the process of individual productivity. I will define this Markov process in this way in order to allow for a persistence in this variable. The main goal of this section is to explain the role that persistence have in the efficiency of the economy. As I showed in the last section, the limit case with infinite persistence will produce a positive effect on productivity. Now, it is not hard to see that in the case with zero persistence, the heterogeneity of the credit limit will not have any effect and the economy will behave (in the aggregate) as if all the entrepreneurs face the same value of the limit credit $\lambda$ given by $E(\lambda)$. Therefore, I would expect a soft transition of the aggregate productivity $Z$ between these two extreme cases. The higher the persistence of the credit limit $\lambda$, the higher the aggregate productivity would be. Given that this generalization has an extremely complex mathematical structure I will only simulate the results in order to reinforce our intuition in this framework. I will begin with the definition of the Markov process of the credit limit $\lambda$, it will be defined in the
same way as Moll (2014) defined the process of talent $z$.

$$d\lambda = \mu_\lambda dt + \sigma_\lambda dW$$

Where $\mu_\lambda$ is the drift term and $\sigma_\lambda$ is the diffusion term. Here I will assume that the diffusion term is independent, ergodic and have a unique stationary distribution. This process might have a correlation with the process of the entrepreneurial talent $z$ but this will be a more complicated mathematical model and the results should be similar.

Now, with this new dynamic system for $\lambda$ I am able to know the dynamics of $\omega$ and find a necessary condition for the steady state.

**Lemma 7.0.1.** Suppose that $\lambda$ and $z$ are Markov processes. Then, the wealth shares $\omega_t(z, \lambda)$ obey the second-order partial differential equation

$$\frac{\partial \omega_t(z, \lambda)}{\partial t} = \left[ s_t(z, \lambda) - \frac{K_t}{K_t} \right] \omega_t(z, \lambda) - \frac{\partial \mu_z(z) \omega_t(z, \lambda)}{\partial z} + \frac{1}{2} \frac{\partial^2 \sigma^2(z) \omega_t(z, \lambda)}{\partial z^2} - \frac{\partial \mu_\lambda(\lambda) \omega_t(z, \lambda)}{\partial \lambda} + \frac{1}{2} \frac{\partial^2 \sigma^2(\lambda) \omega_t(z, \lambda)}{\partial \lambda^2}$$

Therefore, the stationary wealth shares $\omega(z, \lambda)$ obey the second-order ordinary differential equation

$$0 = s(z, \lambda) \omega(z, \lambda) - \frac{\partial \mu_z(z) \omega(z, \lambda)}{\partial z} + \frac{1}{2} \frac{\partial^2 \sigma^2(z) \omega(z, \lambda)}{\partial z^2} - \frac{\partial \mu_\lambda(\lambda) \omega(z, \lambda)}{\partial \lambda} + \frac{1}{2} \frac{\partial^2 \sigma^2(\lambda) \omega(z, \lambda)}{\partial \lambda^2}$$

This result is obtained by using the Kolmogorov forward equation in a multivariate Markov process. Note that if $\mu_\lambda(\lambda)$ and $\sigma_\lambda(\lambda)$ are zero and $\lambda$ is unique for all the en-
entrepreneurs, then \( \omega_t(z, \lambda) \) does not depend on \( \lambda \) and I obtain the original result of (Moll, 2014) which is proposition 6.1.1. Now, the next step to have a close expression of the total factor productivity in terms of the persistence of the process is to find a steady state of the distribution function \( \omega \). Then I use the results of the aggregate variables of the economy and use comparative statics in order to find an increase in the total factor productivity due to an increase in the persistence of the process of credit limit.

### 7.1 Simulations

In this subsection I will observe how the persistence of the \( \lambda \) process affects the productivity through simulations. As I saw earlier, when there is fixed financial constraints, the mean preserving spread of the limit constraint \( \lambda \) produces a positive effect in the total factor productivity of the economy and a huge effect on inequality. Now, with a dynamic financial constraint there should be movement of resources among the population leading to a decrease in inequality and a decrease in total factor productivity. Therefore I could guess that the greater the persistence, the lower the levels of the inequality and productivity.

For this simulations I will assume that both processes follow a Ornstein-Uhlenbeck process i.e.

\[
\begin{align*}
    d \log z & = \frac{-1}{\theta_z} \log z \, dt + \sigma_z \sqrt{\frac{1}{\theta_z}} \, dW \\
    d \log \hat{\lambda} & = \frac{-1}{\theta_{\hat{\lambda}}} \log \hat{\lambda} \, dt + \sigma_{\hat{\lambda}} \sqrt{\frac{1}{\theta_{\hat{\lambda}}}} \, dW
\end{align*}
\]
where \( \lambda = \hat{\lambda} + 1 \) to warrant that \( \lambda > 1 \).

I simulate several economies each one with different persistence parameters. Then I plot the mean of the total factor productivity of each of those economies to observe if there is any pattern. Note that each economy will have the same distribution of lambda, varying only the persistence (or velocity) of lambda across entrepreneurs. I would expect that the higher the persistence parameter, the higher the aggregate productivity.

Figure 7.1: Aggregate productivity in terms of persistence of the credit limit \( \lambda \)

I also plot the mean of productivity of four different economies (represented by black lines). Each economy will have a single and fixed value of credit limit defined as the 50\%, 90\%, 95\% and 99\% percentile of the distribution of the credit limit \( \lambda \) respectively. Note that these percentiles do not depend on the persistence of the process of \( \lambda \). In fact, the marginal distribution function will be equal for each value of the persistence of \( \lambda \), i.e. the persistence
of $\lambda$ just determines the individual transition velocity but not the shape of the distribution.

Finally, I can see in figure 7.1 that there is a positive correlation between the persistence of $\lambda$ and the aggregate productivity of the economy that is consistent with our intuition. However it can be seen that the level of persistence of the variable $\lambda$ should be very high in order to approximate to the case where there is a completely persistence heterogeneity in the credit limit $\lambda$. 
Chapter 8

Conclusion

In this thesis project, I developed a model of heterogeneous borrowing constraints in a financial frictions framework, specifically, the framework of Moll (2014) of financial frictions. Firstly I analyzed the case where a mean preserving spread shock of the credit limit disturbs the economy. In this case, I showed that immediately after the shock the aggregate productivity starts to grow. This positive effect prevails over time and, in the long run, converges to an aggregate productivity of an analogous economy with the highest credit limit for all the entrepreneurs. In contrast, it is shown that when the spread shock decreases the mean of the entrepreneurs’ credit limit, then the total factor productivity is negatively affected immediately after the shock, leading a decrease in total factor productivity in the short run. Nevertheless, under the assumption that some firms have a higher credit limit, this negative effect is counteracted in the long run. Firms with a higher credit limit accumulate all the capital of the economy, and asymptotically the economy behaves like an analogous economy but with an homogeneous higher credit limit, leading to an improve of the aggregate productivity. In both cases it is shown that total factor productivity is increased in the long run. This observation opens the door to welfare improvement government interventions. I argue that if the government could monitor closely some selected group of firms, then the
likelihood of a running-away firm been caught is higher. Then, the banks take this into account and can lend more to this selected group of firms increasing its borrowing constraint. Of course the cost of such a policy could be a lower monitoring for the rest of the firms by the government. Analogously, this should lead to a decrease in the borrowing constraint of the rest of the firms. Nevertheless, given the results of this thesis project, I conclude that this kind of policy will lead to higher productivity in the long run.

It should be emphasized that all these results strongly depend on the constant returns to scale assumption, I use this assumption in order to simplify the computation of the aggregate variables of the economy. In a more general model, the computation of the aggregate variables become extremely complex. It is shown that the heterogeneity in the borrowing constraint directly implies that less firms will produce in the long run. However because of the constant returns to scale assumption, these effects should not disturb the productivity of the economy. In contrast with a decreasing returns to scale production function. Even though, empirical evidence of decreasing returns to scale is present, I argue that the positive effect in total factor productivity still prevails when the selected group of firms with benefits in the credit market is sufficiently large.
Chapter 9

Mathematical appendix

This appendix include all the proofs of this thesis project.

9.1 Proof of proposition 4.0.1

The proof is exactly the same as in Moll (2014):

I now that the firm’s profits are given by

$$\Pi(a, z) = \max_{k,l} \{f(z, k, l) - wl - (r + \delta)k \text{ s.t. } k \leq \lambda a\}$$

Then, using the first order condition with respect to $l$ I find that

$$l^* = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{1-\alpha}} z \ k$$
and plugging it back into the profits function I get

\[ \Pi(a, z) = \max_k \{ z \pi k - (r + \delta) k \text{ s.t. } k \leq \lambda a \} \]

Then, this became a linear maximization problem in \( k \) that leads a solution where \( k \) is either zero or \( \lambda a \). The cutoff \( z \) is determined where the entrepreneurs are indifferent between producing or not. This means that \( \pi(a, z) = 0 \).

\[ \square \]

9.2 Proof of proposition 4.0.2

The proof is exactly the same as in Moll (2014):

I define

\[ A(z, t) = \lambda \max \{ z \pi(t) - r(t) - \delta, 0 \} + r(t) \]

Then I know from 4.0.1 that

\[ \dot{a} = A(z, t) a - c \]

The Bellman equation is then

\[ \rho V(a, z, t) = \max_c \left\{ \log(c) + \frac{1}{d} \mathbb{E}[dV(a, z, t)] \text{ s.t. } da = [A(z, t) - c]dt \right\} \]

Guessing that the value function takes the form \( V(a, z, t) = Bv(z, t) + B \log(a) \), I have that
\[ E[dV(a,z,t)] = \left( \frac{B}{a} \right) da + B E[dv(z,t)] \]

Rewriting the value function, I get

\[ \rho B v(z,t) + \rho B \log(a) = \max c \log(c) + \frac{B}{a} [A(z,t)a - c] + B \frac{1}{dt} E[dv(z,t)] \]

Taking the first-order condition, I obtain that \( c = a/B \). Substituting back into the previous expression, I get:

\[ \rho B v(z,t) + \rho B \log(a) = \log(c) - \log(B) + A(z,t)B - 1 + B \frac{1}{dt} E[dv(z,t)] \]

Now, I can see that \( B = 1/\rho \) so that \( c = \rho a \) and \( \dot{a} = [A(z,t) - \rho]a \) as claimed. Finally, the value function is

\[ V(a,z,t) = [v(z,t) + \log(a)]/\rho \]

where \( v(z,t) \) satisfies

\[ \rho(z,t) = \rho \log(\rho) - \rho + A(z,t) + \frac{1}{dt} E[dv(z,t)] \]
9.3 Proof of proposition 5.2.2

This proof is analogous to the original in Moll (2014). I will omit the index \( t \) for notational simplicity. From lemma 4.0.1 if \( z \geq z \) I get

\[
l(a, z, \lambda) = \left( \frac{\pi}{\alpha} \right)^{\frac{1}{1-\alpha}} \lambda a z
\]

and substituting it in the production function I obtain

\[
y(a, z, \lambda) = (zk)^{\alpha l^{1-\alpha}}
= (z\lambda a)^{\alpha} \frac{\pi}{\alpha} (z\lambda a)^{1-\alpha}
= \frac{\pi}{\alpha} z\lambda a
\]

if \( z \geq z \) and zero otherwise. Then total production is

\[
Y = \int_{z}^{\infty} \int_{1}^{\infty} \int_{0}^{\infty} y(a, z, \lambda) g(a, z, \lambda) \, da \, d\lambda \, dz
= \int_{z}^{\infty} \int_{1}^{\infty} \int_{0}^{\infty} \frac{\pi}{\alpha} \lambda a z \, g(a, z, \lambda) \, da \, d\lambda \, dz
= \frac{\pi}{\alpha} \int_{z}^{\infty} \int_{1}^{\infty} \lambda z \int_{0}^{\infty} a \, g(a, z, \lambda) \, da \, d\lambda \, dz
= \frac{\pi}{\alpha} \int_{z}^{\infty} \int_{1}^{\infty} \lambda z K \, \omega(z, \lambda) \, d\lambda \, dz
= \frac{\pi}{\alpha} K \int_{z}^{\infty} \int_{1}^{\infty} \lambda z \, \omega(z, \lambda) \, d\lambda \, dz
= \frac{\pi}{\alpha} K X
\]

where

\[
X = \int_{z}^{\infty} \int_{1}^{\infty} \lambda z \, \omega(z, \lambda) \, d\lambda \, dz
\]
Analogously

\[
L = \int_0^{\alpha} \int_1^{\alpha} \int_0^{\alpha} l(a, z, \lambda) g(a, z, \lambda) \, da \, d\lambda \, dz
\]

\[
= \int_0^{\alpha} \int_1^{\alpha} \int_0^{\alpha} \left( \frac{\pi}{a} \right) \frac{1}{1-\alpha} \lambda a z \, g(a, z, \lambda) \, da \, d\lambda \, dz
\]

\[
= \left( \frac{\pi}{a} \right) \frac{1}{1-\alpha} \int_0^{\alpha} \int_1^{\alpha} \lambda z \int_0^{\alpha} a \, g(a, z, \lambda) \, da \, d\lambda \, dz
\]

\[
= \left( \frac{\pi}{a} \right) \frac{1}{1-\alpha} \int_0^{\alpha} \int_1^{\alpha} \lambda z K \, \omega(z, \lambda) \, d\lambda \, dz
\]

\[
= \left( \frac{\pi}{a} \right) \frac{1}{1-\alpha} K \int_0^{\alpha} \int_1^{\alpha} \lambda z \omega(z, \lambda) \, d\lambda \, dz
\]

\[
= \left( \frac{\pi}{a} \right) \frac{1}{1-\alpha} K X
\]

if and only if

\[
\pi = \alpha (XK)^{\alpha-1} L^{1-\alpha}
\]

Then, substituting in the total production I obtain

\[
Y = \frac{\alpha (XK)^{\alpha-1} L^{1-\alpha}}{\alpha} K X = X^\alpha K^\alpha L^{1-\alpha} = Z K^\alpha L^{1-\alpha}
\]
where $Z = X^\alpha$ is total factor productivity. Now from the modified equation 5.2.2 and using the clearing credit market condition 5.2.2 it can be seen that

$$\frac{\dot{K}_t}{K_t} = \int_1^\infty \int_0^\infty \left[ \lambda \max\{z \pi(t) - r(t) - \delta, 0\} + r(t) - \rho \right] \omega_t(z, \lambda) \, dz \, d\lambda$$

$$= \int_1^\infty \int_{\hat{z}}^\infty \left[ \lambda(z \pi - r - \delta) + r - \rho \right] \omega(z, \lambda) \, dz \, d\lambda + \int_1^\infty \int_0^z [r - \rho] \omega(z, \lambda) \, dz \, d\lambda$$

$$= \int_1^\infty \int_{\hat{z}}^\infty \left[ \lambda(z \pi - r - \delta) \right] \omega(z, \lambda) \, dz \, d\lambda + r - \rho$$

$$= \pi \int_1^\infty \int_{\hat{z}}^\infty \lambda z \omega_t(z, \lambda) \, dz \, d\lambda - \int_1^\infty \int_{\hat{z}}^\infty \lambda \omega_t(z, \lambda) \, dz \, d\lambda + r - \rho$$

$$= \pi X - [r + \delta] + r - \rho$$

$$= \pi X - (\delta - \rho)$$

and substituting the value of $\pi$ I get

$$\dot{K}_t = \alpha X^\alpha K^\alpha L^{1-\alpha} - (\delta - \rho) K = \alpha Z K^\alpha L^{1-\alpha} - (\delta - \rho) K$$

Now, to obtain an expression for $w$ I substitute the equation of $\pi$ in Lemma 4.0.1

$$\pi = \alpha (X K)^{\alpha-1} L^{1-\alpha}$$

Rearranging yields

$$w = (1 - \alpha) X^\alpha K^\alpha L^\alpha$$

$$= (1 - \alpha) Z K^\alpha L^\alpha$$
Finally to obtain the value of $r$ I substitute the cutoff condition of the Lemma 4.0.1 to get

$$r = \bar{z} \pi - \delta$$

$$= \bar{z} \alpha X^{\alpha-1} K^{\alpha-1} L^{1-\alpha} - \delta$$

$$= \bar{z} \alpha Z^{\alpha-1} K^{\alpha-1} L^{1-\alpha} - \delta$$

9.4 Proof of proposition 6.1.1

The proof is analogous to the original from Moll (2014).

9.5 Proof of lemma 6.1.3

We are interested about how behave $Z$ and $\bar{z}$ in a small neighbourhood around the shock time $t^*$.

Firstly, we will prove that a mean preserving spread will preserve the continuity of $\bar{z}(t)$ and $Z(t)$ at the mean preserving spread shock at $t^*$.

To show this, note that in time $t = t^* - \epsilon$ for an infinitesimal $\epsilon$, the clear market equation of capital warrants

$$\int_{\bar{z}(t)}^{\infty} \lambda \omega_t(\lambda, z)dz = 1$$
but, using the mean spread definition, which is

\[ \bar{\lambda} \omega_t(\bar{\lambda}, z) = \lambda_H \omega_{t*}(\lambda_H, z) + \lambda_L \omega_{t*}(\lambda_L, z) \quad \text{for every } z \]

then at time of the shock \(t^*\) we get

\[ \int_{\underline{z}(t)}^{\infty} \lambda_H \omega_{t*}(\lambda_H, z) + \lambda_L \omega_{t*}(\lambda_L, z) \, dz = 1 \]

leading that

\[ \underline{z}(t^* - \epsilon) = \underline{z}(t^*) \]

i.e. the function \(\underline{z}\) is continuous in time.

Secondly we will prove the continuity of \(Z\).

by definition, if \(t = t^* - \epsilon\)

\[ Z(t) = \left( \int_{\underline{z}(t)}^{\infty} \bar{\lambda} \omega_t(z, \lambda) \, dz \right)^{\alpha} \]

and, using again the definition of mean preserving spread and the continuity of \(\underline{z}\) I get

\[ Z(t) = \left( \int_{\underline{z}(t^*)}^{\infty} z \left[ \lambda_H \omega_{t*}(\lambda_H, z) + \lambda_L \omega_{t*}(\lambda_L, z) \right] \, dz \right)^{\alpha} = Z(t^*) \]

therefore \(Z\) is also continuous.

Now I will prove that both variables are increasing in time. To prove it, observe that for
each \( t, \hat{z} \) is defined by the equation

\[
\int_{\hat{z}(t)}^{\infty} \int_{1}^{\infty} \lambda \omega_t(\lambda, z) \, dz \, d\lambda = 1.
\]

Deriving with respect to time I get

\[
-\dot{\hat{z}}(t) \int_{1}^{\infty} \lambda \omega_t(\hat{z}(t), \lambda) \, d\lambda + \int_{\hat{z}(t)}^{\infty} \int_{1}^{\infty} \lambda \dot{\omega}_t(z, \lambda) \, d\lambda \, dz = 0
\]

and rearranging

\[
\dot{\hat{z}}(t) = \frac{\int_{\hat{z}(t)}^{\infty} \int_{1}^{\infty} \lambda \omega_t(z, \lambda) \, d\lambda \, dz}{\int_{1}^{\infty} \lambda \omega_t(\hat{z}(t), \lambda) \, d\lambda}
\]

Then \( \dot{\hat{z}}(t) > 0 \) is in this case equivalent to

\[
\int_{\hat{z}(t)}^{\infty} \int_{1}^{\infty} \lambda \dot{\omega}_t(z, \lambda) \, d\lambda \, dz > 0
\]

Now, if \( t^* \) denotes the time of the mean preserving shock and \( z \geq \hat{z} \). I will prove, using the linearity of \( S \) and that \( \lambda \) is independent of \( z \), that

\[
\int_{1}^{\infty} \lambda \dot{\omega}_{t^*}(z, \lambda) \, d\lambda > 0 \quad (9.1)
\]
Let’s proceed.

\[ \int_{\lambda} \omega_t^*(z, \lambda) = \int_{\lambda} \lambda \left( s(z, \lambda) \omega_t^*(z, \lambda) - \frac{\partial \omega_t^*(z, \lambda) \mu(z)}{\partial z} + \frac{1}{2} \frac{\partial^2 \omega_t^*(z, \lambda) \sigma^2(z)}{\partial z^2} \right) d\lambda \]

\[ = \int_{\lambda} \lambda s(z, \lambda) \omega_t^*(z) f(\lambda) d\lambda - \int_{\lambda} \lambda \frac{\partial \omega_t^*(z) \mu(z)}{\partial z} \left( \int_{\lambda} \lambda f(\lambda) d\lambda \right) \]

\[ + \int_{\lambda} \frac{1}{2} \frac{\partial^2 \omega_t^*(z) \sigma^2(z)}{\partial z^2} \left( \int_{\lambda} \lambda f(\lambda) d\lambda \right) \]

\[ = \omega_t^*(z) \int_{\lambda} \lambda s(z, \lambda) f(\lambda) d\lambda - \frac{\partial \omega_t^*(z) \mu(z)}{\partial z} \left( \int_{\lambda} \lambda f(\lambda) d\lambda \right) \]

\[ + \frac{1}{2} \frac{\partial^2 \omega_t^*(z) \sigma^2(z)}{\partial z^2} \left( \int_{\lambda} \lambda f(\lambda) d\lambda \right) \]

where \( f(\lambda) \) is the distribution of the mean preserving spread shock of \( \lambda \) and \( \omega_t^*(z) \) is the marginal density. Now, given that for each \( z \) the variables \( \lambda \) and \( S(\lambda, z) \) are positively correlated then

\[ \int_{\lambda} \lambda s(z, \lambda) f(\lambda) d\lambda = E[ \lambda s(z, \lambda) | z ] = E[ \lambda ] E[ s(z, \lambda) | z ] = \bar{\lambda} s(z, \bar{\lambda}) \]

and then

\[ \int_{\lambda} \lambda \omega_t^*(z, \lambda) = \omega_t^*(z) \bar{\lambda} s(z, \bar{\lambda}) - \frac{\partial \omega_t^*(z) \mu(z)}{\partial z} \bar{\lambda} + \frac{1}{2} \frac{\partial^2 \omega_t^*(z) \sigma^2(z)}{\partial z^2} \bar{\lambda} \]

\[ = \bar{\lambda} \left( \omega_t^*(z) s(z, \bar{\lambda}) - \frac{\partial \omega_t^*(z) \mu(z)}{\partial z} + \frac{1}{2} \frac{\partial^2 \omega_t^*(z) \sigma^2(z)}{\partial z^2} \right) \]

\[ = \bar{\lambda} \left( \omega_t(z) s(z, \bar{\lambda}) - \frac{\partial \omega_t(z) \mu(z)}{\partial z} + \frac{1}{2} \frac{\partial^2 \omega_t(z) \sigma^2(z)}{\partial z^2} \right) \]

\[ = \bar{\lambda} \omega_t(z) \]

\[ = 0 \]
I can see then that (9.1) holds and therefore $\dot{\bar{z}}(t^*) > 0$. i.e. the mean preserving spread shock is leading to a higher productivity cutoff of the firms.

Now to prove that $\dot{Z}(t) > 0$ rememering that $Z(t) = [X(t)]^\alpha$. with

$$X(t) = \int_{\bar{z}(t)}^{\infty} \int_{1}^{\infty} \lambda \omega(z, \lambda) d\lambda \, dz$$

Therefore it is enough to show that $\dot{X}(t^*) > 0$. Deriving the previous expression with respect to time and substituting $\dot{\bar{z}}(t)$ I can see that

$$\dot{X}(t) = \int_{\bar{z}(t)}^{\infty} \int_{1}^{\infty} \lambda \, z \, \dot{\omega}(z, \lambda) \, d\lambda \, dz - \dot{\bar{z}}(t) \int_{1}^{\infty} \lambda \, \omega(\bar{z}, \lambda) \, d\lambda$$

Then in this case it is enough to prove that for each $z > \bar{z}$,

$$\int_{1}^{\infty} \lambda \, \dot{\omega}_t(\lambda) \, d\lambda \, dz > 0$$

what I have already proven in (9.1). Therefore I conclude that $\dot{Z}(t^*) > 0$, i.e. the mean preserving spread shock is producing a positive effect in total factor productivity. 

\[\square\]
9.6 Proof of Lemma 6.1.5

Note that

\[ \frac{K_H}{K_L} = \frac{\int_0^\infty \omega(z, \lambda_H) \, dz}{\int_0^\infty \omega(z, \lambda_L) \, dz} \]

Then

\[
\left( \frac{\dot{K}_H}{\dot{K}_L} \right) = \frac{\int_0^\infty \dot{\omega}(z, \lambda_H) \, dz \int_0^\infty \omega(z, \lambda_L) \, dz - \int_0^\infty \dot{\omega}(z, \lambda_L) \, dz \int_0^\infty \omega(z, \lambda_H) \, dz}{\left( \int_0^\infty \omega(z, \lambda_L) \, dz \right)^2} \\
= \frac{\int_0^\infty \omega_H \, dz \frac{K_H}{K_L} - \int_0^\infty \omega_L \, dz \frac{K_H}{K_L}}{K_L^2} \\
= \left[ \frac{\int_0^\infty \omega_H \, dz \, K_L - \int_0^\infty \omega_L \, dz \, K_L}{K_L^2} \right] K
\]

Note also that from proposition 6.1.1 I have the following expression

\[
\int_0^\infty \dot{\omega}(z, \lambda_i) \, dz = \int_0^\infty \left[ s(z, \lambda_i) - \dot{K} K^{-1} \right] \omega(z, \lambda_i) \, dz \\
= \text{Total increase in wealth} - \left[ \int_0^\infty \frac{\partial}{\partial z} \mu(z) \omega(z, \lambda_i) \, dz + \frac{1}{2} \int_0^\infty \frac{\partial^2}{\partial z^2} \sigma^2(z) \omega(z, \lambda_i) \, dz \right] \text{Movement of wealth among } z
\]

There, the movement of wealth within each group of entrepreneurs with credit limit \( \lambda_i \) and given type of \( z \) is equal to the last two terms of the right side. These aggregate movements of wealth should be zero because these dynamics should not affect the total amount of wealth that the group with limit \( \lambda_i \) has. It can be proved that these terms are zero using the fundamental theorem of calculus and noting that \( \omega \) is zero and flat enough in the limits where \( z \to 0 \) or \( z \to \infty \). Then I get

\[
\int_0^\infty \dot{\omega}(z, \lambda_i) \, dz = \int_0^\infty \dot{\omega}(z, \lambda_i) s(z, \lambda_i) \, dz - \frac{\dot{K} K_i}{K^2}
\]
where \( i \) denotes \( H \) or \( L \) and therefore

\[
\left( \frac{\dot{K}_H}{K_L} \right) > 0 \quad \iff \quad \int_0^\infty \hat{\omega}(z, \lambda_H) \, dz \, K_L - \int_0^\infty \hat{\omega}(z, \lambda_L) \, dz \, K_H > 0
\]

\[
\iff \quad \int_0^\infty s(z, \lambda_H)\omega(z, \lambda_H) \, dz \, K_L > \int_0^\infty s(z, \lambda_L)\omega(z, \lambda_H) \, dz \, K_H
\]

\[
\iff \quad \frac{\int_0^\infty s(z, \lambda_H)\omega(z, \lambda_H) \, dz}{K_H} > \frac{\int_0^\infty s(z, \lambda_L)\omega(z, \lambda_H) \, dz}{K_L}
\]

\[
\iff \quad E_\omega[s(\lambda, z) \mid \lambda = \lambda_H] > E_\omega[s(\lambda, z) \mid \lambda = \lambda_L]
\]

\( \square \)

### 9.7 Proof of Lemma 6.1.6

Suppose that there is a steady state with prices \( r^*, w^* \) and cutoff \( \bar{z}^* \) as given. Now, if I denote \( g = \frac{K_i}{K} \) and \( \omega^*(z) \) is the steady state distribution of a certain group of entrepreneurs, then \( g \) should satisfy

\[
[s(z, \lambda) - g] \omega^*(z) - \frac{\partial}{\partial z} \mu(z) \omega^*(z) + \frac{1}{2} \frac{\partial^2 \sigma^2(z) \omega^*(z)}{\partial z^2} = 0 \quad \text{for all } z
\]

Now, integrating with respect to the talent of all the individuals I get

\[
g = \int_0^\infty s(z, \lambda) \omega^*(z) \, dz
\]

and deriving with respect to \( \lambda \)

\[
\frac{\partial g}{\partial \lambda} = \int_0^\infty \frac{\partial}{\partial \lambda} s(z, \lambda) \omega^*(z) \, dz > 0.
\]

Then \( g \) is an increasing function of \( \lambda \). Then, noting that \( \lambda_H > \lambda_L \) I conclude that
\[ \frac{K_H}{K_H} > \frac{K_L}{K_L}. \]

### 9.8 Proof of Lemma 6.1.7

Suppose that a steady-state economy with only one credit limit \( \bar{\lambda} \) has a simple spread shock with two new limits \( \lambda_L \) and \( \lambda_H \) at time \( t^* \).

Firstly note that in the limit

\[
\lim_{t \to \infty} \frac{\partial \log \left( \frac{K_L}{K_H} \right)}{\partial t} = \lim_{t \to \infty} \frac{\partial \log(K_H)}{\partial t} - \frac{\partial \log(K_L)}{\partial t} = \lim_{t \to \infty} \left[ \frac{\dot{K}_H}{K_H} - \frac{\dot{K}_L}{K_L} \right]
\]

where lemma (6.1.6) warrants that \( \lim_{t \to \infty} \left[ \frac{\dot{K}_H}{K_H} - \frac{\dot{K}_L}{K_L} \right] \) is a negative constant. Therefore, I conclude that

\[
\lim_{t \to \infty} \log \left( \frac{K_L}{K_H} \right) = -\infty.
\]

Then

\[
\lim_{t \to \infty} \frac{\dot{K}_L}{K_H} = 0
\]

and finally

\[
\lim_{t \to \infty} \frac{K_L + \dot{K}_H}{K_H} = 1.
\]

Therefore, in the long run \( K_L = \epsilon K_H \) for an infinitesimal \( \epsilon \). This means that \( \dot{K}_H \approx \dot{K} \) and \( \dot{K}_H \approx \dot{K} \).

Now, using lemma (6.1.1)
\[
\dot{w}(z, \lambda_H) \, dz = \left[ s(z, \lambda_H) - \frac{\dot{K}}{K} \right] \omega(z, \lambda_H) \, dz - \frac{\partial}{\partial z} \mu(z) \omega(z, \lambda_H) \, dz + \frac{1}{2} \frac{\partial^2}{\partial z^2} \sigma^2(z) \omega(z, \lambda_H) \, dz \\
\approx \left[ s(z, \lambda_H) - \frac{\dot{K}_H}{K_H} \right] \omega(z, \lambda_i) \, dz - \frac{\partial}{\partial z} \mu(z) \omega(z, \lambda_i) \, dz + \frac{1}{2} \frac{\partial^2}{\partial z^2} \sigma^2(z) \omega(z, \lambda_H) \, dz
\]

Where the approximation could be done as good as needed. Then, since all the capital is owned by the \( H \) group of entrepreneurs, this means that

\[
\omega(z, \lambda_H) \approx 1 \text{ i.e. } K \approx K_H.
\]

All the prices \( r, w \) and all the aggregate variables depend just on the values of \( \omega(z, \lambda_H) \) and are independent from the values of \( \omega(z, \lambda_L) \).

Then, this dynamical system could be approximated by the following system:

\[
\dot{w}(z) \, dz = \left[ s(z, \lambda_H) - \frac{\dot{K}}{K} \right] \omega(z) \, dz - \frac{\partial}{\partial z} \mu(z) \omega(z) \, dz + \frac{1}{2} \frac{\partial^2}{\partial z^2} \sigma^2(z) \omega(z) \, dz
\]

which is the dynamical system of an economy with just the credit limit \( \lambda_H \) for all the entrepreneurs.
9.9 Proof of the lemma 7.0.1

The proof is analogous to the proof of Moll (2014). It is the same just using the Kolmogorov forward equation in a multivariate Markov process.
References


