CENTRO DE INVESTIGACIÓN Y DOCENCIA ECONÓMICAS, A.C.

TOPICS IN MACROECONOMETRICS

TESINA

QUE PARA OBTENER EL TÍTULO DE

LICENCIADO EN ECONOMÍA

PRESENTA

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MÉXICO, D.F. AGOSTO, 2016

A mi Padre, a mi Madre y a mi Hermana.
A su amor y fortaleza.

Agradecimientos

Estudiar Economía en el Centro de Investigación y Docencia Económicas representó un gran cambio estructural en mi vida. Cada momento de mi carrera universitaria representó un fuerte reto, retos a los que no me enfrenté sólo. Por ello, considero que es necesario el reconocimiento a todos aquellos y aquellas que estuvieron conmigo en esta etapa.

Le agradezco a la Prepa 6 de la UNAM por brindarme el primer acercamiento al mundo académico, por los maravillosos momentos que me hizo pasar y, sobre todo, por haberme dado a la familia que ahí conocí. Amigos que, a pesar de muchos factores, nunca me dejaron sólo y de quienes recibí un fuerte y constante apoyo durante toda mi carrera.

Le agradezco a Carlos, Jorge, Charlie, Germán, Jesús, Lily, Belén, Yuni, Zabina, Luis, Balam, Rebeca, Iván, Gonzalo, Alejandra, Jonathan, Santiago, Alexandra, Roger, Gustavo, Diana y a David por su amistad a lo largo de la carrera. De todos ustedes me llevo grandes momentos y aprendizajes.

Le agradezco a cada profesor del Centro de Investigación y Docencia Económicas por la entrega y pasión con que buscan transmitir su conocimiento. En especial, reconozco el fuerte compromiso que la División de Economía tiene con la investigación, reflejado en sus excepcionales trabajos y su preocupación por el desarrollo, académico y personal, de los que hemos tenido la oportunidad de ser sus estudiantes.

Le agradezco a mi seminarista, el Dr. Kaniska Dam y a mi lector, el Dr. Arturo Antón, por su paciencia, buena disposición y excelente guía en el desarrollo de mi tesina. Por sus extraordinarias clases y por todo lo que aprendí de ellos.

Particularmente, le agradezco profundamente al Dr. Daniel Ventosa-Santaulària por el gran apoyo que recibí de su parte desde el día en que supe que era mi tutor académico. Le agradezco por todas las oportunidades que me dio, que fueron desde las pláticas mensuales para hablar de mi estancia en el CIDE hasta la oportunidad de aprender como su estudiante y asistente de investigación. Le agradezco por permitirme hacer mi tesina bajo su dirección y encontrar en las Series de Tiempo una de mis más grandes pasiones.

Por último y a modo de resaltar su importancia, le agradezco inmensamente a mi padre Guillermo, a mi madre Martha y a mi hermana Alejandra por nunca haberme dejado sólo, por haber soportado la carrera conmigo y siempre apoyarme y guiarme en todo el camino. Por haber creído en mí en todo momento, por haberme levantado en todas las caídas que he tenido. A ustedes dedico por completo mi trabajo, esfuerzo que ha resultado ser uno de los más importantes y satisfactorios en mi vida.

Abstract

The present study analyzes two topics in macroeconometrics: the first one is on the power of the Dickey and Fuller (1979) test and, the second one, about inference issues because of aggregation and smoothing methods in macroeconomics.

The first topic is focused on the analysis of the asymptotic properties of the Dickey-Fuller test under the alternative hypothesis of stationarity. Punctually, we studied the power of such test, this is, the probability of not making a type-II error (accepting the null hypothesis when it is false). We analyzed the limit behavior of the Dickey-Fuller test under the alternative hypothesis of stationarity. Through a Monte-Carlo experiment, we were also able to study its finite sample behavior as well as its dynamics when the sample size grows. Then, we proposed reporting the power in a similar way in which the size is reported to obtain the relevant properties of the t-ratios of the estimated parameters of the test.

The second topic analyzes the aggregation and smoothing methods on macroeconomics. Under standard conditions, i.e., under stationarity, the social scientist would be able to draw inference using test statistics. The t-ratios associated with estimated parameters, for example, are widely used in applied economics. Asymptotically, this statistic converges to the normal distribution under the null hypothesis. This means that the practitioner would use critical values derived from a standard normal distribution. However, using Phillips'(1986) theoretical setting, we show that the distribution of the t-ratio associated with the coefficient for a regression that uses aggregated variables does not converge to a standard normal, but remains centered at zero; its tails are narrower than those of the standard normal. Hence, the critical values traditionally used are incorrect in the inferential analysis of the regression: The econometrician may over-reject the null hypothesis and the inference is misleading.

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Chapter 1

General introduction

Econometrics plays a fundamental role in economic analysis and has been an important tool in macroeconomics research of (lineal) relationships amongst variables. Since Legendre (1805) developed the *Ordinary Least Squares (OLS)*[1] and Galton (1888) discovered and developed the concept of *correlation*,[2] there have been many studies whose objectives have been to propose new econometric procedures and to improve those already existed. Thereby, the present study analyzed two topics in macroeconometrics: the first one is on the power of the Dickey and Fuller (1979) test and, the second one, about inference issues because of aggregation and smoothing methods in macroeconomics.

The first topic is focused on the analysis of the asymptotic properties of the *Dickey-Fuller* test under the alternative hypothesis of stationarity. Punctually, we studied the power of such test, this is, the probability of not making a type-II error (accepting the null hypothesis when it is false). Traditionally, practitioners (of statistics) focus on the size (or significance level) of the t-ratios associated with the estimated parameters of the test and acknowledge implicitly that they can not avoid error type-I and error type-II simultaneously. Nevertheless, the power has

[1]The method was used to calculate the path of comets; there is a well-known discussion about the original author of the technique, as Friedrich Gauss also claimed for the authorship of the method.

[2]The idea was applied on anthropometric data in order to find the power of the heritage among parents and their offspring. For an excellent review, see Stigler (1989).

been a property ignored in the report of the t-ratios.

In order to improve such information, we studied the limit behavior of the *Dickey-Fuller* test under the alternative hypothesis of stationarity. Through a Monte-Carlo experiment, we were also able to study its finite sample behavior as well as its dynamics when the sample size grows. Then, we proposed reporting the power in a similar way in which the size is reported to obtain the relevant properties of the t-ratios of the estimated parameters of the test.

The second topic analyzes the aggregation and smoothing methods on macroeconomics. The aggregation/smoothing of variables, mainly in macroeconomics, may affect statistical inference under standard assumptions in econometric analysis. Data aggregation such as smoothing methods, is a common exercise in macroeconomic studies, as it allows to reduce the problems that may arise when we are working with variables that fluctuate and suffer cyclical movements; such problems hinder the understanding of the dynamics and nature of the data. Examples of this can be found, *inter alia*, in Feldstein and Horioka (1980), Mehra and Prescott (1985), Barro and Sala-i Martin (1992) and Fama and French (2002). However, to the best of our knowledge, there is scarce research in econometrics on the effect of aggregation in standard statistical inference.

Under standard conditions, i.e., under stationarity, the social scientist would be able to draw inference using test statistics. The t-ratios associated with estimated parameters, for example, are widely used in applied economics. Asymptotically, this statistic converges to the normal distribution under the null hypothesis. This means that the practitioner would use critical values derived from a standard normal distribution. However, we found that aggregation, such as the one obtained by moving or simple average, affects the asymptotic distribution of the test statistic under the null. Therefore, the standard normal critical values are incorrect and inference is misleading.

Using Phillips' (1986) theoretical setting, we show that the distribution of the t-ratio associated with the coefficient for a regression that uses aggregated variables does not converge to a standard normal, but remains centered at zero; its tails are narrower than those of the standard normal. Hence, the critical values traditionally used are incorrect in the inferential analysis of the regression: The econometrician may over-reject the null hypothesis.

It is worthwhile mentioning that we found the correct critical values for the inferential analysis with aggregated stationary data through a Monte-Carlo experiment. We propose the use of these new critical values as a solution over the inference troubles that may arise through the use of smoothed variables.

Chapter 2

The power of the Dickey-Fuller test

2.1 Literature review

There is a vast literature on unit roots on time-series econometrics. It is necessary to start our recap from the 1970s. Granger and Newbold (1974) showed the serious problems that arise when the practitioners are working with nonstationary data, i.e., random variables whose first moments (mean and variance) depend on time. The authors showed how a regression may provide nonsense inference, wrongly rejecting the null hypothesis of non-relationship between two variables generated by two completely independent processes. Hence, the authors cogently argued that the practitioner should study the properties and the nature of the variables under analysis.

Shortly after, Dickey and Fuller (1979) proposed a test capable of identifying the presence of unit root on the data. They used the properties of the data-generating process (DGP, hereafter) $y_t = \rho y_{t-1} + e_t$, with a constant initial condition (y_0) and $e_t \sim \mathcal{N}(0, \sigma^2)$. Dickey and Fuller explained both the stationary nature of the process when $|\rho| < 1$ and the nonstationary when $|\rho| \geq 1$. For the second case, if the coefficient equals the unity, the variance of y_t will grow by a rate of T ($T\sigma^2$). If ρ is strictly higher than the unity, the variance will grow in an exponential rhythm in the same time in which t is increasing.

Nelson and Plosser (1982) contributed in an exceptional way on this test studying the nature of the principal macroeconomic series of the United States. In their analysis, the authors searched for evidence about the precise behavior of the variables, i.e, they studied if the series were *stationary fluctuations around a deterministic trend* or were *nonstationary processes*. Nelson and Plosser showed that the behavior of most of the macro-series could better be described as a unit root, advertising of the high hazard of draw nonsense inference if it was not treated with the due care.

Phillips (1986) proposed an elegant theoretical framework to understand the phenomenon of spurious regression identified by Granger and Newbold (1974). Such a framework included the *Brownian motion*, *the Functional Central Limit Theorem* and the *Continuous Mapping Theorem*, Phillips employed an asymptotic theory capable of describing the behavior of a non stationary processes. Thus, the author showed that the t statistic associated with the estimated parameter of a spurious regression diverges at $T^{\frac{1}{2}}$ rate, such that, as sample size grows, the null hypothesis of no-correlation between two variables (which are generated by two independent processes) would be eventually rejected.

Furthermore, Phillips (1987) presented other fundamental results in the asymptotic analysis of regressions under stochastic nonstationarity. Using the same tools mentioned above, the author discovered the statistics properties of the t-test and key parameters. In that sense, Phillips' results underpin the relevance of the Dickey and Fuller (1979). In order to improve the quality of inference yielded by the *Dickey-Fuller* test, Phillips and Perron (1988), proposed a new test to infer the presence of unit root in the variable. The main advantage of the Phillips-Perron test is that the autocorrelation structure is estimated nonparametrically.

Shortly after, Kwiatkowski, Phillips, Schmidt, and Shin (1992) proposed a test which null hypothesis is the stationary around a deterministic trend and the alternative the nonstationarity of the data. Moreover, Elliott, Rothenberg, and Stock (1996) proposed a *"family of tests"* which included an asymptotically optimal point to detect the unit root. Their modification over the *Dickey-Fuller* test improved it when there is an unknown mean or trend.

2.2 Data-generating processes and specification

2.2.1 Data-generating processes

To analyze the power of the Dickey-Fuller test this is, the probability of correctly rejecting the null hypothesis when it is false (or, in other words, not making the type II error), we studied its properties under the alternative hypothesis ($\mathcal{H}_\alpha$), i.e., stationary data. Hence, the DGPs will be the following:

$$y_t = u_{yt}, \tag{2.1}$$

$$y_t = \mu_y + u_{yt}, \tag{2.2}$$

$$y_t = \mu_y + \delta_y t + u_{yt}, \tag{2.3}$$

where $u_{yt} \sim \text{iid}\mathcal{N}(0, \sigma_y^2)$; equation 2.1 refers to the simplest possible DGP, where there is neither constant term nor deterministic trend; equation 2.2 allows the constant term but not the deterministic trend. Finally, equation 2.3 allows both the constant term and the deterministic trend.

2.2.2 Specification

To apply the *Dickey-Fuller test* on her data, the practitioner may use the following auxiliary regressions:

$$
\begin{aligned}
y_t &= \alpha y_{t-1} + \varepsilon_t, &\quad (2.4)\\
y_t &= \beta + \alpha y_{t-1} + \varepsilon_t, &\quad (2.5)\\
y_t &= \beta + \delta t + \alpha y_{t-1} + \varepsilon_t, &\quad (2.6)
\end{aligned}
$$

where $\varepsilon_t \sim \text{iid}\mathcal{N}(0,\sigma_y^2)$; subtracting y_{t-1} in each side of the equations 2.4, 2.5 and 2.6, we obtain respectively:

$$
\begin{aligned}
y_t - y_{t-1} &= \alpha y_{t-1} - y_{t-1} + \varepsilon_t\\
\Delta y_t &= (\alpha - 1) y_{t-1} + \varepsilon_t\\
\Delta y_t &= \gamma y_{t-1} + \varepsilon_t \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
y_t - y_{t-1} &= \beta + \alpha y_{t-1} - y_{t-1} + \varepsilon_t\\
\Delta y_t &= \beta + (\alpha - 1) y_{t-1} + \varepsilon_t\\
\Delta y_t &= \beta + \gamma y_{t-1} + \varepsilon_t \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
y_t - y_{t-1} &= \beta + \delta t + \alpha y_{t-1} - y_{t-1} + \varepsilon_t\\
\Delta y_t &= \beta + \delta t + (\alpha - 1) y_{t-1} + \varepsilon_t\\
\Delta y_t &= \beta + \delta t + \gamma y_{t-1} + \varepsilon_t \qquad (2.9)
\end{aligned}
$$

According with the above transformations over the regression, we worked with the hypothesis posed by Dickey and Fuller (1979), this is:

$$\mathcal{H}_0: \quad \hat{\gamma} = 0 \quad \text{Unit Root}$$

$$\mathcal{H}_\alpha: \quad \hat{\gamma} < 0 \quad \text{Stationarity}$$

With the above specifications, we were able to analyze the properties of the t statistic associated with the $\hat{\gamma}$ coefficient in 2.7, 2.8 and 2.9. The next section presents the results from our analysis.

2.3 Asymptotic results

According to the analysis under the alternative hypothesis of the Dickey-Fuller test, we obtained the limit distribution for each DGP. For it, we used the *Brownian motion*,[1] *the Functional Central Limit Theorem*[2] and the *Continuous Mapping Theorem*[3] in order to find the asymptotic expressions of the t-ratio under the alternative. Through a thorough analysis, we searched for the first and second orders of convergence to be able to represent the correct convergence of distribution expression for all the DGPs presented above. Most of the t-ratios associated with the estimated parameters convergence in distribution when $\sqrt{T}$ is added. When we do that, we find that the t-ratios converge to a normal distribution. That distribution expressions help us in the study of the power of the test, in the sense that we have an approximation of its behavior as the sample size grows. All our results are original and we present it from theorem 2.3.1 to theorem 2.3.6. In addition, we include the proof of theorem 2.3.1 on Appendix A in order to show our method to solve and find the convergence distributions under the alternative hypothesis of the Dickey-Fuller test; the following theorems have a similar procedure.

[1] That allow us to represent the continuous-time process as $T \to \infty$

[2] Used to obtain a result of normality with nonstationary random variables

[3] Allow us to transform with finite sample expressions. For more details, see Hamilton (1994)

- **The simplest DGP & specification.**

Theorem 2.3.1. *Let y_t be generated as equation 2.1, estimate the specification 2.7. Then, as $T \to \infty$:*

$$t_\gamma + \sqrt{T} \xrightarrow{D} \omega_y(1),$$

where $\omega_y(\cdot)$ is a standard Brownian motion. Note that $\omega_y(1)$ is just a standard normal.

- **Simplest DGP & specification with constant.**

Theorem 2.3.2. *Let y_t be generated as equation 2.1, estimate the specification 2.8. Then, as $T \to \infty$:*

$$t_\gamma + \sqrt{T} \xrightarrow{D} \omega_y(1),$$

where $\omega_y(\cdot)$ is a standard Brownian motion. Note that $\omega_y(1)$ is just a standard normal.

- **DGP & specification with constant.**

Theorem 2.3.3. *Let y_t be generated as equation 2.2, estimate the specification 2.8. Then, as $T \to \infty$:*

$$t_\gamma + \sqrt{T} \xrightarrow{D} \omega_y(1),$$

where $\omega_y(\cdot)$ is a standard Brownian motion. Note that $\omega_y(1)$ is just a standard normal.

- **DGP with constant & specification with constant and trend.**

Theorem 2.3.4. *Let y_t be generated as equation 2.2, estimate the specification 2.9. Then, as $T \to \infty$:*

$$t_\gamma + \sqrt{T} \xrightarrow{D} \omega_y(1),$$

where $\omega_y(\cdot)$ is a standard Brownian motion. Note that $\omega_y(1)$ is just a standard normal.

- **DGP with constant and trend & specification with constant and trend.**

Theorem 2.3.5. *Let y_t be generated as equation 2.3, estimate the specification 2.9. Then, as $T \to \infty$:*

$$t_\gamma + \sqrt{T} \xrightarrow{D} \omega_y(1),$$

where $\omega_y(\cdot)$ is a standard Brownian motion. Note that $\omega_y(1)$ is just a standard normal.

- **DGP with correlation & specification with constant.**

Theorem 2.3.6. *Let y_t be generated as $y_t = \varepsilon_{y,t}$, where $\varepsilon_{y,t} = \psi(L)u_{yt}$ and u_{yt} is an i.i.d. sequence, estimate by OLS $\Delta y_t = \alpha + \gamma y_{t-1} + \varepsilon_t$. Then, as $T \to \infty$:*

$$\begin{aligned}
\underbrace{t_\gamma + \sqrt{T}}_{t_\gamma^{mod}} &= O_p(T^{\frac{1}{2}}) \\
T^{-\frac{1}{2}} t_\gamma^{mod} &\xrightarrow{P} \frac{\sqrt{\gamma_0 + \gamma_1} - \sqrt{\gamma_0 - \gamma_1}}{\sqrt{\gamma_0 + \gamma_1}},
\end{aligned}$$

where $\gamma_0 = E(u_t^2)$ and $\gamma_1 = E(u_t u_{t-1})$.

2.4 Monte-Carlo experiments

Once we got the convergences of the t statistic under the specific DGPs, we were able to analyze the behavior of the alternative hypothesis ($\mathcal{H}_\alpha$) for different sample sizes. The Figure 2.1 shows such behavior, which is consistent with all the DGPs described in the previous section since all of them converge to the same expression, except the DGP which allows correlation. We can observe the leftward displacement of the distribution under $\mathcal{H}_\alpha$ as the sample size (T) grows. This allows us to asymptotically ensure the rejection of the null hypothesis when this is false, showing that the power of the test is higher as we increase the sample size. Approximately, this occurs when the sample size has fifty observations.

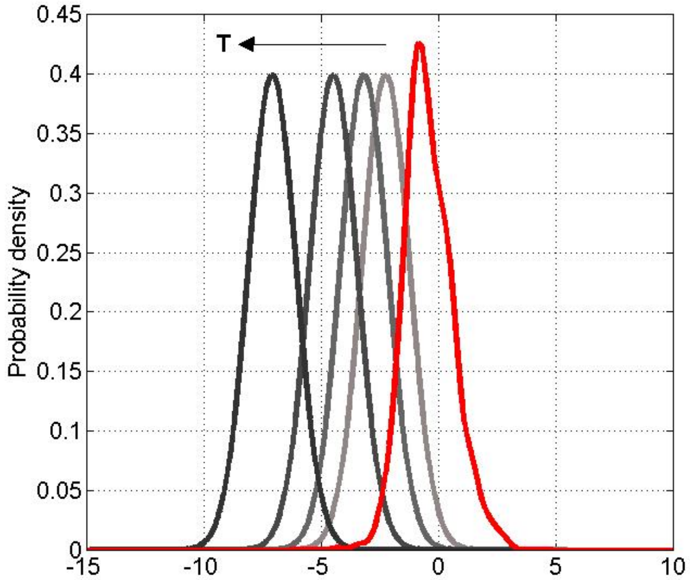

Figure 2.1: The figure shows the behavior of the t-ratio associated with the estimated parameter under both the null and the alternative hypothesis of the D-F test. The red solid line represents the estimated density of t_γ under $\mathcal{H}_0$ and the following gray solid lines represent the densities of t_γ under $\mathcal{H}_\alpha$ (stationarity) for $T = (5, 10, 20, 50)$ and 10,000 replications.

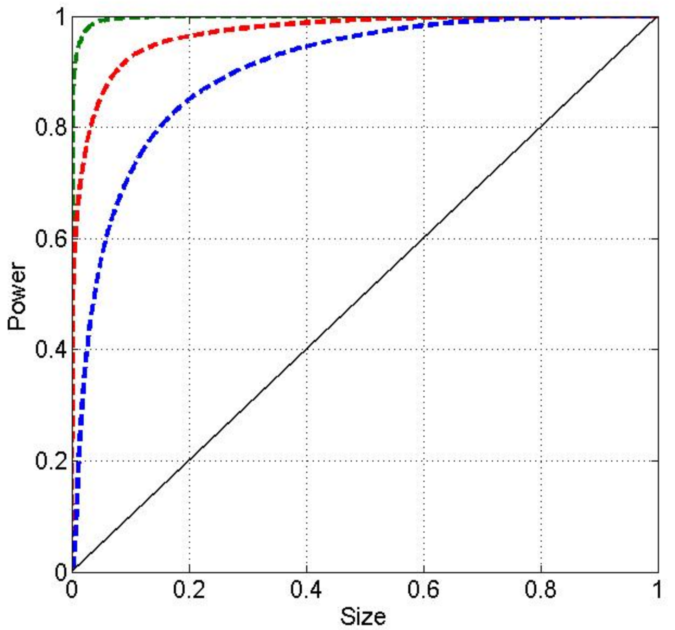

Figure 2.2: The figure shows the Size-Power trade-off of the D-F test. The color dashed lines represent the trade-off between the Size and the Power for $T = (10, 20, 50)$. As we can see, the growth of the sample size improves both the size (decreases) and the power (increases) of the t-ratio.

The aforementioned results allow us to emphasize the neglected importance of the power in any statistical testing procedure. We, therefore, propose reporting the power of a test in a similar way to the level; this is, we propose using the † symbol which would play the role of the classic asterisk (*) when size is reported. Analogously, one dagger (†) would represent a power of 90%, two daggers (††) a power of 95% and three daggers († † †) a power of 99%. Thus, we could obtain all the relevant information carried by a test statistic: (1) the probability of making a type-I error (size of the test) and (2) the probability of not making a type-II error (the power of the test). Then, the most desirable test statistic would take the form:

$$\boxed{t^{***,\dagger\dagger\dagger}.}$$

The later t-ratio would be read as follows: the null hypothesis would be rejected at the 1% level, whilst the power of the test is superior to 99%. This new notation would allow the practitioner to acknowledge at first sight what level/power trade-off is she enduring.

2.5 Empirical illustration

We applied our methodology over the Nelson & Plosser extended data,[4] which contains the principal macroeconomic series of the USA. We include a quick review about the nature of the macroeconomic series, made it by several studies on the past. The review is included in Tables 2.2 and 2.3.

As we can see from Tables 2.2 and 2.3, most of the variables present a nonstationary behavior. We studied the Nelson and Plosser data set building the t-ratios associated with the estimated parameters of the D-F test. Consistently with the conclusions of the studies described in Tables 2.2 and 2.3, we have rejected the null hypothesis of stationarity of the data. In addition and in order to describe the power of the t statistic associated for each variable using the Dickey & Fuller test, we have written the levels of significance and power according with our suggest.

[4]The data set includes the original data which was presented by Nelson and Plosser (1982) plus an extension of the data presented by Koop and Steel (1994). The database can be found at the website of Daniel Ventosa-Santaulària: `http://www.ventosa-santaularia.com/NP_database.html`

Table 2.1 shows our results.

Variable	Constant	Constant & trend	Sample
Real GNP	$0.6548^{\dagger\dagger\dagger}$	$-0.2250^{\dagger\dagger\dagger}$	[1909-1988]
Nominal GNP	$0.2459^{\dagger\dagger\dagger}$	$-1.7096^{\dagger\dagger\dagger}$	[1909-1988]
Real per capita GNP	$1.5504^{\dagger\dagger\dagger}$	$-1.7184^{\dagger\dagger\dagger}$	[1909-1988]
Industrial Production	$1.4331^{\dagger\dagger\dagger}$	$-0.1373^{\dagger\dagger\dagger}$	[1860-1988]
Total Employment	$2.6293^{\dagger\dagger\dagger}$	$-0.8526^{\dagger\dagger\dagger}$	[1890-1988]
Total Unemployment Rate	$-2.8068^{***\dagger\dagger\dagger}$	$-2.9857^{*\dagger\dagger\dagger}$	[1890-1988]
GNP Deflactor	$0.2484^{\dagger\dagger\dagger}$	$-0.5237^{\dagger\dagger\dagger}$	[1889-1988]
Consumer Price Index	$1.0858^{\dagger\dagger\dagger}$	$-0.7961^{\dagger\dagger\dagger}$	[1860-1988]
Nominal wages	$0.0858^{\dagger\dagger\dagger}$	$-1.5776^{\dagger\dagger\dagger}$	[1900-1988]
Real wages	$0.7554^{\dagger\dagger\dagger}$	$-1.4787^{\dagger\dagger\dagger}$	[1900-1988]
Money Stock	$-0.3386^{\dagger\dagger\dagger}$	$-1.3974^{\dagger\dagger\dagger}$	[1989-1988]
Velocity of Money	$0.374^{\dagger\dagger\dagger}$	$-0.4230^{\dagger\dagger\dagger}$	[1869-1988]
Bond Yield	$-0.1370^{\dagger\dagger\dagger}$	$-1.2283^{\dagger\dagger\dagger}$	[1900-1988]
Stock Prices	$1.8576^{\dagger\dagger\dagger}$	$-0.3282^{\dagger\dagger\dagger}$	[1871-1988]

Table 2.1: Application over the N&P extended dataset.

We can see that most of the *Dickey-Fuller* tests present a high power, this is, in all cases there is a low probability of committing a type-II Error; this is showed by the three daggers in all of them. However, the significance of the t statistics are lower, i.e., there is a high probability or, at least, higher than 10% of making the type I error (rejecting the null when it is true).

The illustration above has an important empirical implication. Now, the t-ratios gives to the practitioner a more complete information about the probability of the statistical errors that she may commit. On the one hand, the t statistics presented in our empirical illustration show that we should not reject the null hypothesis of non stationary data, with a probability of commit the I-type error over the 10 percent. On the second hand, the t-ratios give the additional information that the probability of making a II-type error is less than the 1 percent. For example, from Table 2.1, the practitioner will note, at first sight, that she could not reject the null hypothesis of unit root for the *Nominal GNP* and that she is facing a probability less than the 1 percent of wrongly rejecting the alternative hypothesis of the test. Such information prevents the practitioner about the implications of her t-ratios and may improve the decisions of the economist about the nature of her data.

2.6 Concluding remarks

We analyzed the properties of the *Dickey-Fuller* test under the alternative in order to show the power of such test. Once we found the asymptotic distributions expressions of the t statistic associated with the estimated parameters, we were able to postulate the theorems which summarize our results for the combinations of data-generating processes and the specifications described above. We then studied the distributions of the t-statistics, showing the displacement of the alternative in a leftward direction of the distribution of the $\mathcal{H}_{\alpha}$ as the sample size increases; this property makes the test more powerful as we increase the sample size.

Importantly, we used the *Dickey-Fuller* test and focused on the neglected importance of its power. There is scarce information about this statistical property, even thought its importance on the inferential analysis is fundamental in order to avoid erroneous conclusions. We, therefore, proposed to report the power of the test in a similar way in which the size is reported. This would allow the practitioner to have both, information about the probability of committing type-I error and information about the probability of committing type-II error. We consider appropriate providing complete information in any statistical test if a proper analysis is to be conducted.

Table 2.2: The table shows the several studies that have been done around the Nelson and Plosser data set. The table includes the author or authors of the study, the year in which the study was published, the technique/procedure used, the number of breaks on the data found and the conclusion of the study. The conclusion *DS* refers to *difference stationary* (i.e., the data needs to be differentiated to be stationary)), *FI* refers to *fractionally integrated* data and textitTS refers to *trend stationary* (the trend must be estimated and removed from the data) processes.

	Paper	Year	N&P data set	Model/Procedure	Breaks	Conclusion
1	Nelson and Plosser	1982	Original	ADF, constant and trend	0	DS
2	Perron	1988	Original	Phillips Perron	0	DS
3	Perron	1989	Original	Modified ADF with exogenous break	1	TS
4	DeJong y Whiteman	1991	Original	Bayesian modelling	0	TS
5	Stock	1991	Original	Confidence intervals of ADF estimates	0	I
6	KPSS	1992	Original	KPSS test	0	I
7	ZA	1992	Original	Modified ADF with endogenous break	1	DS
8	Crato and Rothman	1994	Extended	ARFIMA estimation	0	DS
9	Lucas	1995	Extended	DF-t via HBP estimator (robust to outliers)	0	TS
10	Li	1995	Original	Recursive UR test	1	DS
				Reverse recursive UR test	1	I
				Rolling UR test	1	I
				Sequential UR test	1	DS
11	Hans Franses and Kleibergen	1996	Original	Forecasting criteria MSPE and MAPE		
				Forecast horizon 12	0	DS
				Forecast horizon 18	0	DS
				Forecast horizon 36	0	TS
				Multistep ahead forecast	0	TS
				One-step ahead, rolling regression	0	DS
12	Gil-Alaña and Robinson	1997	Extended	Robinson 1994 (LM)		
				U white noise	0	DS
				U white noise+ constant	0	DS-FI
				U white noise+ constant+trend	0	DS-FI
				U AR(k)	0	FI-DS
				U bloomfield exponential	0	FI
13	Papell and Prodan	1997	Original	Restricted structural change hypothesis	1	TS

Table 2.3: The table shows the several studies that have been done around the Nelson and Plosser data set. The table includes the author or authors of the study, the year in which the study was published, the technique/procedure used, the number of breaks on the data found and the conclusion of the study. The conclusion *DS* refers to *difference stationary* (i.e., the data needs to be differentiated to be stationary), *FI* refers to *fractionally integrated* data and textitTS refers to *trend stationary* (the trend must be estimated and removed from the data) processes.

	Paper	Year	N&P data set	Model/Procedure	Breaks	Conclusion
14	Perron	1997	Original	Perron Endogenous break	1	I
15	Lumsdaine y Papell	1997	Original	ZA with 2 breaks under Ha	1	TS
16	Kapetanios	2002	Extended	ZA with up to 5 breaks	1	TS
17	Kilian y Ohanian	2002	Original	ZA - State-Space modelling of breaks	1	DS
18	Andreou y Spanos	2003	Original	Critics to NP and Perron over model adequacy	1	TS
19	Sen	2004	Extended	ZA, Perron and Murray and Zivot	1	TS
20	Carrion-i-Silvestre and Sansó	2006	Original	ADF Joint with breaks	1	DS
21	Lee and Strazicich	2003	Original	Their test	1	DS
22	Narayan and Bopp	2010	Original	ADF with 2 breaks		
				M1	1	DS
				M2	1	TS
23	Pascalau	2010		EL and BEL tests, robust to breaks		
			Original	BEL	1	TS
				EL	1	DS
			Extended	BEL	1	TS
				EL	1	TS
24	Ventosa and Gómez	2010	Extended	Focus on drifts	1	TS
25	Darné and Charles	2011	Extended	Focus on outliers, Elliot et al, Ng and Perron		
				DF_GLS	1	DS
				Ng and Perron	1	DS
26	Alexeev and Maynard	2012	Extended	Non parametric level crossing random walk test	1	DS
27	Charles and Darné	2012	Extended	Focus on outliers, ADF-type tests		
				Intervention model	1	TS
				Robust QML	1	TS
28	Mills	2013	Original	Lambdagram	0	DS
29	Grassi and Proietti	2014	Original	Bayesian modelling	0	I

Chapter 3

Inference under data aggregation in empirical macroeconomics

3.1 Literature review

Data aggregation (such as in smoothing methods) is a common exercise in macroeconomic studies, as it allows to reduce the problems that may arise when we are working with variables with fluctuations and cyclical movements. Such problems hinder the understanding of the dynamics and nature of the data. Examples of this can be found, *inter alia*, in Feldstein and Horioka (1980), Mehra and Prescott (1985), Barro and Sala-i Martin (1992) and Fama and French (2002).

There are, however, few studies on the effect of aggregation and smoothing methods on statistical inference. Mundlak (1961) treated this problem in models with distributed lags. Zellner and Montmarquette (1971) adverted about some econometric problems that may arise using aggregated data. According with the authors, there are four main problems through the aggregation, on their words: *"(a) lower precision of estimation and prediction, (b) lower power for tests, (c) inability to make short-run forecasts and (d) a reduction of the probability of discovering new* *hypotheses about short-run behavior from data"*.[1] Zellner and Montmarquette analyzed such problems with a simple econometric model.

However, the asymptotic properties of an aggregated process have not been thoroughly studied. The present study obtained the asymptotic distributions of the t-ratios associated with the estimated parameter for a regression which uses aggregated data. This allows us to detect some nontrivial issues when drawing inference throughout a t-ratio.

3.2 Data-generating processes and specification

3.2.1 Data-generating processes

In order to make the correct analysis, we used different DGPs whose were aggregated through two methods, described in the next section. We started our analysis with the simplest DGPs, represented as a white noise. Then, we were adding some especial econometric features to make our analysis with more complicated DGPs; the most complicated DGP used was the one which included *drift* and follow the behavior of a *unit root*. Furthermore, we included a cointegrated DGP to complete the study. Therefore, we worked with the following DGPs:

$$z_t = u_{zt}, \tag{3.1}$$

$$z_t = \mu_z + u_{zt}, \tag{3.2}$$

$$z_t = z_{t-1} + u_{zt}, \tag{3.3}$$

$$z_t = \mu_z + z_{t-1} + u_{zt}, \tag{3.4}$$

[1] See Zellner and Montmarquette (1971), page 335

$$x_t = x_{t-1} + u_{xt}, \quad (3.5)$$

$$y_t = \mu_y + \beta_y x_t + u_{yt}, \quad (3.6)$$

where $z = y,x$ and $u_{zt} \sim \mathcal{N}(0, \sigma_z^2)$.

3.2.2 Specification

The analysis included two aggregated methods, the *moving* and the *simple average*. We used those methods because they are the most common smoothing techniques around macroeconomics. For example, the first technique is referring to the average which take the three first observations and create a new one observation; its second observation will be the average of the second to fourth observations, losing, in total, the extremes observations of the data. The second technique is referring to the average of the first three observations, being the second new observation the average of the fourth to sixth observations; in this case, we are losing two-thirds of the sample. Then, we can generalize the methods that we used as follows:

$$z_t^{*a} = \frac{z_1 + z_2 + \ldots + z_k}{k}, \frac{z_2 + z_3 + \ldots + z_{k+1}}{k}, \ldots, \frac{z_{t-(k-1)} + z_{t-(k-2)} + \ldots + z_t}{k}, \quad (3.7)$$

$$z_t^{*b} = \frac{z_1 + z_2 + \ldots + z_k}{k}, \frac{z_{k+1} + z_{k+2} + \ldots + z_{2k}}{k}, \ldots, \frac{z_{t-(k-1)} + z_{t-(k-2)} + \ldots + z_t}{k}, \quad (3.8)$$

where equation 3.7 refers to Moving Average aggregation, equation 3.8 to Simple Average, $z = x, y$ and k is the order of the aggregation.

Once we have aggregated the DGP we tested the processes through the equation:

$$y_t^* = \alpha + \beta x_t^* + \varepsilon_t \tag{3.9}$$

We analyzed the convergence distributions of the t statistics, which takes the following form:

$$t_\beta = \frac{\hat{\beta}}{\sqrt{\hat{\sigma}_\beta^2}}, \tag{3.10}$$

where $\hat{\sigma}_\beta^2 = \hat{\sigma}^2 (X'X)_{22}^{-1}$.

3.3 Asymptotic results

The motivation of the analysis described above is to make an approximation of the behavior of the convergence distribution of the t-ratios associated with the estimated parameters when we use some aggregation method and compare them when we used it in normal terms. Once we made the appropriate analyzes according to the DGP and the specification described above, we obtain the asymptotic distributions of the statistics associated with the estimated parameters in order to show the convergence of this statistics when $T \to \infty$. All our results are original and we present it from Theorem 3.3.1 to Theorem 3.3.5. In addition, we include the proof of Theorem 3.3.3 on Appendix B in order to show our method to solve and find the convergence distributions of the expressions for the DGPs under data aggregation.

Theorem 3.3.1. *Let x_t and y_t be generated by equation 3.1 and aggregate them using 3.7 and 3.8, denote this as y_t^{*a}, y_t^{*b}, x_t^{*a} and x_t^{*b}, respectively. Estimate 3.9 by OLS. Then, as $T \to \infty$:*

Using 3.7 (Moving Average):

$$T^{-\frac{1}{2}}\hat{\alpha} \quad \xrightarrow{D} \quad \frac{\sigma_y\left(\int \omega_x(r,k)\int \omega_x(r,k)\omega_y(r,k) - \int \omega_x(r,k)^2 \int \omega_y(r,k)\right)}{(\int \omega_x(r,k))^2 - \int \omega_x(r,k)^2},$$

$$\hat{\beta} \quad \xrightarrow{D} \quad \frac{\sigma_y(\int \omega_x(r,k)\int \omega_y(r,k) - \int \omega_x(r,k)\omega_y(r,k))}{(\int \omega_x(r,k))^2 - \int \omega_x(r,k)^2},$$

$$T^{-\frac{1}{2}}t_\alpha \quad \xrightarrow{D} \quad \frac{\int \omega_x(r,k)\int \omega_x(r,k)\omega_y(r,k) - \int \omega_x(r,k)^2 \int \omega_y(r,k)}{\sqrt{((\int \omega_x(r,k)\omega_y(r,k))^2 - \Gamma_1 + \Gamma_2)\,(\int \omega_x(r,k)^2)}},$$

$$T^{-\frac{1}{2}}t_\beta \quad \xrightarrow{D} \quad -\frac{\int \omega_x(r,k)\int \omega_y(r,k) - \int \omega_x(r,k)\omega_y(r,k)}{\sqrt{((\int \omega_x(r,k)\omega_y(r,k))^2 - \Gamma_1 + \Gamma_2)\,(\int \omega_x(r,k)^2)}},$$

$$R^2 \quad \xrightarrow{D} \quad 1 + \frac{\left((\int \omega_x(r,k)\omega_y(r,k))^2 - \Gamma_1 + \Gamma_2\right)\left(\int \omega_x(r,k)^2\right)}{((\int \omega_x(r,k))^2 - \int \omega_x(r,k)^2)\,((\int \omega_x(r,k))^2 - \int \omega_x(r,k)^2)},$$

$$T^{-1}\mathcal{F} \quad \xrightarrow{D} \quad -\frac{\Gamma_3 + \left((\int \omega_x(r,k)\omega_y(r,k))^2 - \Gamma_1 + \Gamma_2\right)}{((\int \omega_x(r,k)\omega_y(r,k))^2 - \Gamma_1 + \Gamma_2)}.$$

Using 3.8 (Simple Average):

$$T^{-\frac{1}{2}}\hat{\alpha} \quad \xrightarrow{D} \quad \frac{\sigma_y\left(\int \omega_x(r)\int \omega_x(r)\omega_y(r) - \int \omega_x(r)^2 \int \omega_y(r)\right)}{(\int \omega_x(r))^2 - \int \omega_x(r)^2},$$

$$\hat{\beta} \quad \xrightarrow{D} \quad \frac{\sigma_y(\int \omega_x(r)\int \omega_y(r) - \int \omega_x(r)\omega_y(r))}{(\int \omega_x(r))^2 - \int \omega_x(r)^2},$$

$$T^{-\frac{1}{2}}t_\alpha \quad \xrightarrow{D} \quad \frac{\int \omega_x(r)\int \omega_x(r)\omega_y(r) - \int \omega_x(r)^2 \int \omega_y(r)}{\sqrt{((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2)\,(\int \omega_x(r)^2)}},$$

$$T^{-\frac{1}{2}} t_{\beta} \xrightarrow{D} -\frac{\int \omega_x(r) \int \omega_y(r) - \int \omega_x(r)\omega_y(r)}{\sqrt{((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2)\left(\int \omega_x(r)^2\right)}},$$

$$R^2 \xrightarrow{D} 1 + \frac{\left((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2\right)\left(\int \omega_x(r)^2\right)}{((\int \omega_x(r))^2 - \int \omega_x(r)^2)\left((\int \omega_x(r))^2 - \int \omega_x(r)^2\right)},$$

$$T^{-1}\mathcal{F} \xrightarrow{D} -\frac{\Gamma_3 + \left((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2\right)}{((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2)}.$$

Where $\int \omega_z(r)$ refers to the continuous transformation for $\sum z_{t-1}$ (for $z = x, y$) which grows at rate $O_p(T^{\frac{3}{2}})$ and $\int \omega_z(r,k)$ refers to the continuous transformation for the long horizon sum of the same expression. Furthermore, Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, Γ_6, Γ_7 and Γ_8 are functions of *Brownian motion* and DGP parameters and we defined them in Appendix C. We will use the expressions above in the following theorems.

The theorem above is describing the velocity of convergence of the different parameters. As we can see, the α estimated diverges at $\sqrt{T}$ or is $O_p(T^{\frac{1}{2}})$. The β estimated is $O_p(T^0)$ or $O_p(1)$. Both of them, the t-ratios for the estimated parameters diverges at $\sqrt{T}$. The R-squared is $O_p(1)$ and, finally, the $\mathcal{F}$ diverges at T.

Theorem 3.3.2. *Let x_t and y_t be generated by equation 3.2 and aggregate them using 3.7 and 3.8, denote this as y_t^{*a}, y_t^{*b}, x_t^{*a} and x_t^{*b}, respectively. Estimate 3.9 by OLS. Then, as $T \to \infty$:*

Using 3.7 (Moving Average) and 3.8 (Simple Average):

$$\hat{\alpha} \xrightarrow{P} \mu_y$$

$$\hat{\beta} \xrightarrow{D} \frac{\sigma_y(\int \omega_x(r) \int \omega_y(r) - \int \omega_x(r)\omega_y(r))}{(\int \omega_x(r))^2 - \int \omega_x(r)^2},$$

$$T^{-\frac{1}{2}} t_\alpha \xrightarrow{P} \frac{\mu_y}{\sqrt{\sigma_y^2 - \mu_y^2}},$$

$$T^{-\frac{1}{2}} t_\beta \xrightarrow{D} -\frac{\int \omega_x(r) \int \omega_y(r) - \int \omega_x(r)\omega_y(r)}{\sqrt{((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2)(\int \omega_x(r)^2)}},$$

$$R^2 \xrightarrow{D} 1 + \frac{\left((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2\right)\left(\int \omega_x(r)^2\right)}{((\int \omega_x(r))^2 - \int \omega_x(r)^2)((\int \omega_x(r))^2 - \int \omega_x(r)^2)},$$

$$T^{-1}\mathcal{F} \xrightarrow{D} -\frac{\Gamma_3 + \left((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2\right)}{((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2)},$$

The theorem above is describing the velocity of convergence of the different parameters. As we can see, the α and β are $O_p(1)$. Both of them, the t-ratios for the estimated parameters diverges at $\sqrt{T}$. The R-squared is $O_p(1)$ and, finally, the $\mathcal{F}$ diverges at T.

Theorem 3.3.3. *Let x_t and y_t be generated by equation 3.3 and aggregate them using 3.7 and 3.8, denote this as y_t^{*a}, y_t^{*b}, x_t^{*a} and x_t^{*b}, respectively. Estimate 3.9 by OLS. Then, as $T \to \infty$ and if y_t follows aggregation:*

Using 3.7 (Moving Average) and 3.8 (Simple Average):

$$T^{-\frac{1}{2}}\hat{\alpha} \xrightarrow{D} \frac{\sigma_y\left(\int \omega_x(r) \int \omega_x(r)\omega_y(r) - \int \omega_x(r)^2 \int \omega_y(r)\right)}{(\int \omega_x(r))^2 - \int \omega_x(r)^2},$$

$$\hat{\beta} \xrightarrow{D} \frac{\sigma_y(\int \omega_x(r)\int \omega_y(r) - \int \omega_x(r)\omega_y(r))}{(\int \omega_x(r))^2 - \int \omega_x(r)^2},$$

$$T^{-\frac{1}{2}} t_\alpha \xrightarrow{D} \frac{\int \omega_x(r)\int \omega_x(r)\omega_y(r) - \int \omega_x(r)^2 \int \omega_y(r)}{\sqrt{((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2)(\int \omega_x(r)^2)}},$$

$$T^{-\frac{1}{2}} t_\beta \xrightarrow{D} -\frac{\int \omega_x(r)\int \omega_y(r) - \int \omega_x(r)\omega_y(r)}{\sqrt{((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2)(\int \omega_x(r)^2)}},$$

$$R^2 \xrightarrow{D} 1 + \frac{\left((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2\right)\left(\int \omega_x(r)^2\right)}{((\int \omega_x(r))^2 - \int \omega_x(r)^2)((\int \omega_x(r))^2 - \int \omega_x(r)^2)},$$

$$T^{-1}\mathcal{F} \xrightarrow{D} -\frac{\Gamma_3 + \left((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2\right)}{((\int \omega_x(r)\omega_y(r))^2 - \Gamma_1 + \Gamma_2)}.$$

The theorem above is describing the velocity of convergence of the different parameters. As we can see, the α estimated diverges at $\sqrt{T}$ or is $O_p(T^{\frac{1}{2}})$. The β estimated is $O_p(T^0)$ or $O_p(1)$. Both of them, the t-ratios for the estimated parameters diverges at $\sqrt{T}$. The R-squared is $O_p(1)$ and, finally, the $\mathcal{F}$ diverges at T.

Theorem 3.3.4. *Let x_t and y_t be generated by equation 3.4 and aggregate them using 3.7 and 3.8, denote this as y_t^{*a}, y_t^{*b}, x_t^{*a} and x_t^{*b}, respectively. Estimate 3.9 by OLS. Then, as $T \to \infty$:*

Using 3.7 (Moving Average) and 3.8 (Simple Average):

$$T^{-\frac{1}{2}}\hat{\alpha} \xrightarrow{D} \frac{2\left(2\mu_x \int \omega_y(r) + 3\mu_y \int r\omega_x(r) - 3\mu_x \int r\omega_y(r) - 2\mu_y \int \omega_x(r)\right)}{\mu_x},$$

$$\hat{\beta} \xrightarrow{P} \frac{\mu_y}{\mu_x},$$

$$T^{-\frac{1}{2}} t_{\alpha} \xrightarrow{D} \frac{2(2\mu_x \int \omega_y(r) + 3\mu_y \int \omega_x(r) - 3\mu_x \int r\omega_y(r) - 2\mu_y \int \omega_x(r))}{\sqrt{4(\Gamma_4 - 3\mu_x \int r\omega_y(r) - \Gamma_5 + \Gamma_7 - \Gamma_8 + \Gamma_6)}},$$

$$T^{-\frac{1}{2}} t_{\beta} \xrightarrow{D} \frac{\mu_y \mu_x}{\sqrt{12(\Gamma_4 - \Gamma_5 - 4(\mu_x^2)(\int \omega_y(r)^2) + (\mu_x^2) \int \omega_y(r)^2 - \Gamma_8 + \Gamma_6)}},$$

$$R^2 \xrightarrow{D} 1 - \frac{12(\Gamma_4 - \Gamma_5 - 4(\mu_x^2)(\int \omega_y(r)^2) + (\mu_x^2) \int \omega_y(r)^2 - \Gamma_8 + \Gamma_6)}{(\mu_y^2)(\mu_x^2)T},$$

$$T^{-2}\mathcal{F} \xrightarrow{D} \frac{\mu_x^2 \mu_y^2}{12(\Gamma_4 - \Gamma_5 - 4(\mu_x^2)(\int \omega_y(r)^2) + (\mu_x^2) \int \omega_y(r)^2 - \Gamma_8 + \Gamma_6)}.$$

The theorem above is describing the velocity of convergence of the different parameters. As we can see, the α estimated diverges at $\sqrt{T}$ or is $O_p(T^{\frac{1}{2}})$. The β estimated is $O_p(T^0)$ or $O_p(1)$. Both of them, the t-ratios for the estimated parameters diverges at $\sqrt{T}$. The R-squared is $O_p(1)$ and, finally, the $\mathcal{F}$ diverges at T^2.

Theorem 3.3.5. *Let x_t and y_t be generated by equation 3.5 and 3.6, respectively, and aggregate them using 3.7 and 3.8, denote this as y_t^{*a}, y_t^{*b}, x_t^{*a} and x_t^{*b}, respectively. Estimate 3.9 by OLS. Then, as $T \to \infty$:*

Using 3.7 (Moving Average) and 3.8 (Simple Average):

$$\hat{\alpha} \xrightarrow{P} \mu_y,$$

$$\hat{\beta} \xrightarrow{P} \beta_y,$$

$$T^{-\frac{1}{2}} t_\alpha \xrightarrow{D} \frac{\mu_y \sqrt{\int \omega_x(r)^2 - \left(\int \omega_x(r)\right)^2}}{\sigma_y \sqrt{\int \omega_x(r)^2}},$$

$$T^{-1} t_\beta \xrightarrow{D} \frac{\beta_y \sigma_x \sqrt{\int \omega_x(r)^2 - \left(\int \omega_x(r)\right)^2}}{\sigma_y},$$

$$R^2 \xrightarrow{D} 1 - \frac{\sigma_y^2}{\sigma_x^2 T^2 \left(\beta_y^2 \int \omega_x(r)^2 - \left(\int \omega_x(r)\right)^2\right)},$$

$$T^{-2} \mathcal{F} \xrightarrow{D} \frac{\sigma_x^2 T^2 \left(\beta_y^2 \int \omega_x(r)^2 - \left(\int \omega_x(r)\right)^2\right) - \sigma_y^2}{\sigma_x},$$

The theorem above is describing the velocity of convergence of the different parameters. As we can see, the α and β are $O_p(1)$. The t-ratio for the estimated parameters diverges at $\sqrt{T}$ and T, respectively. The R-squared is $O_p(1)$ and, finally, the $\mathcal{F}$ diverges at T^2.

The asymptotic distributions of the t statistics associated with the estimated parameters are not suffering any change under a nonstationary DGP.[2] Nevertheless, such statistics have a lot of distortions when we are working with stationary data. Therefore, we will focus our analysis on the distributions with this latter DGP. According with the results postulated above, we show the plot of the asymptotic distribution of the statistics working with stationary data. In order to visualize the effect of each aggregation method, we compare, in the respectively case, the normal-standard distribution.[3] Figures 3.1, 3.2, 3.3 and 3.4 show those differences.

[2]We include, in Appendix D the asymptotic distributions of the t-ratios associated with the estimated parameter for the nonstationary DGPs in order to prove this conclusion.

[3]In addition, we include, in Appendix E, the *Matlab* code that we made to verify and to model Theorem 3.3.3 The other theorems have a very similar code.

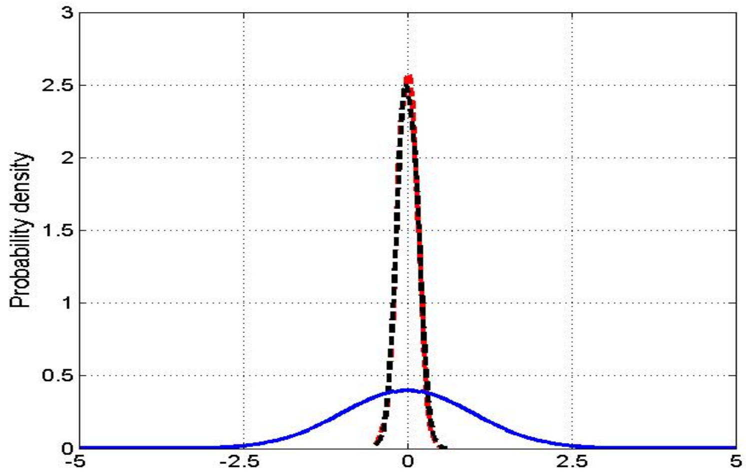

Figure 3.1: **Distribution of the t-ratio for the simplest stationary DGP (represented by 3.1) aggregated by Moving Average.** The blue solid line represents the probability density of the standard Normal distribution , i.e., the DGP with form 3.1 without any aggregation procedure. The red dashed line shows the probability density for the t-ratio associated with the estimated parameters of the simplest stationary DGP for $T = 1{,}000$ and 10,000 replications. We verified our results contrasting our asymptotic distribution versus the distribution obtained from a Monte-Carlo simulation (represented by the black dashed line). Since the distributions are the same, a fact which validates our asymptotic results.

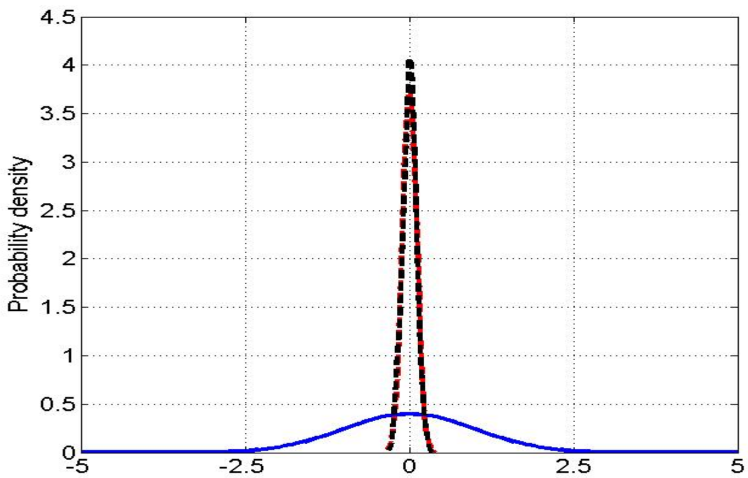

Figure 3.2: **Distribution of the t-ratio for the simplest stationary DGP (represented by 3.1) aggregated by Simple Average.** The blue solid line represents the probability density of the standard Normal distribution , i.e., the DGP with form 3.1 without any aggregation procedure. The red dashed line shows the probability density for the t-ratio associated with the estimated parameters of the simplest stationary DGP for $T = 1{,}000$ and 10,000 replications. We verified our results contrasting our asymptotic distribution versus the distribution obtained from a Monte-Carlo simulation (represented by the black dashed line).

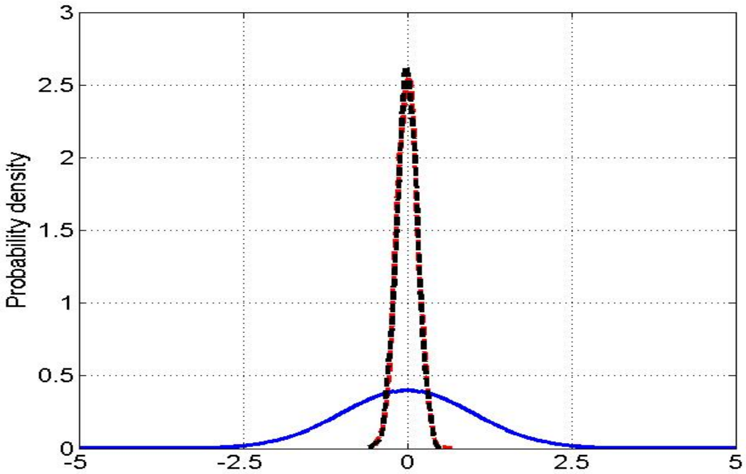

Figure 3.3: **Distribution of the t-ratio for the stationary DGP which includes a constant (represented by 3.2) and is aggregated by Moving Average.** The blue solid line represents the probability density of the standard Normal distribution , i.e., the DGP with form 3.2 without any aggregation procedure. The red dashed line shows the probability density for the t-ratio associated with the estimated parameters of the stationary DGP which includes a constant term for $T = 1,000$ and 10,000 replications. We verified our results contrasting our asymptotic distribution versus the distribution obtained from a Monte-Carlo simulation (represented by the black dashed line).

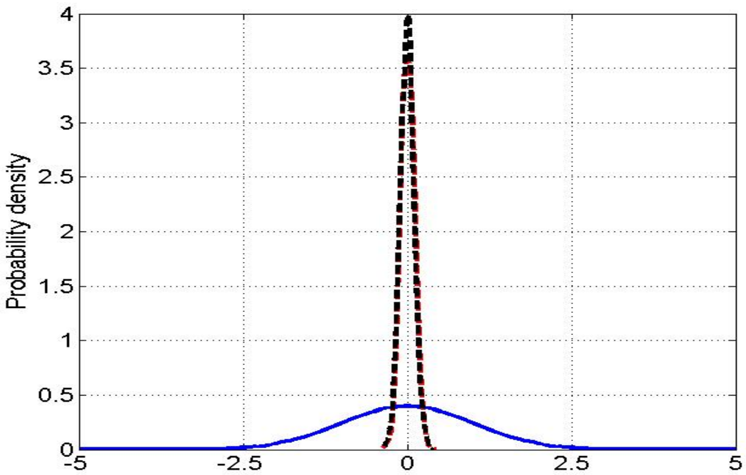

Figure 3.4: **Distribution of the t-ratio for the stationary DGP which includes a constant (represented by 3.2) and is aggregated by Simple Average.** The blue solid line represents the probability density of the standard Normal distribution , i.e., the DGP with form 3.2 without any aggregation procedure. The red dashed line shows the probability density for the t-ratio associated with the estimated parameters of the stationary DGP which includes a constant term for $T = 1,000$ and 10,000 replications. We verified our results contrasting our asymptotic distribution versus the distribution obtained from a Monte-Carlo simulation (represented by the black dashed line).

As we can see from Figures 3.1, 3.2, 3.3 and 3.4, the distributions obtained under aggregation are far from similar from the normal standard distribution. Therefore, the standard critical values will not be the correct to make inferential analysis. Since the practitioner take inferential decisions according with such critical values, she could be making an over-rejecting of her null hypothesis. Through a Monte-Carlo simulation, we found the critical values which could be used when we aggregate stationary data. In order to improve our inferential analysis, the economist may use it to obtain better results. Table 3.1 presents our critical values for the stationary process.

	Stationary process					
	One tail			Two tails		
T \ Size	1%	5%	10%	1%	5%	10%
50	0.5318	0.3662	0.2822	0.5912	0.4417	0.3662
100	0.3577	0.2458	0.1914	0.3942	0.2962	0.2458
200	0.2469	0.1719	0.1334	0.2727	0.2056	0.1719
500	0.1525	0.1078	0.0836	0.1693	0.1294	0.1078
1000	0.1068	0.0756	0.0588	0.1184	0.0903	0.0756

Table 3.1: Critical values for stationary DGP with form 3.1 and a specific order of aggregation k. The values were obtained through 100,000 replications.

3.4 Monte-Carlo experiment

Once we obtained the appropriate critical values to evaluate the statistic associated with the estimated parameter of a test which is using aggregated data, we simulate, through a Monte-Carlo experiment, the behavior of the rejecting rate of this statistic at a 5% level, using both the standard normal and our critical values. Table 3.2 presents our results. On the one hand, we can see that, as we increase the sample size, the rejecting rate is near to 5%; on the other hand, the rejecting rate is almost zero when we use the standard critical values, regardless of the great sample size.

DGP	T=50		T=100		T=250		T=500	
$z_t = u_{zt}$	New CV	Standard CV	New CV	Standard CV	New CV	Standard CV	New CV	Standard CV
$\sigma^2 = 0.5$	0.0449	<0.0001	0.0481	<0.0001	0.0641	<0.0001	0.0532	<0.0001
$\sigma^2 = 1$	0.0428	<0.0001	0.0488	<0.0001	0.0616	<0.0001	0.0501	<0.0001
$\sigma^2 = 2.5$	0.0480	<0.0001	0.0506	<0.0001	0.0665	<0.0001	0.0542	<0.0001

Table 3.2: Rejection rates for a 5% significance level using: (i) our critical values; (ii) standard critical values. 10,000 replications.

3.5 Concluding remarks

We showed that aggregation and smoothing methods, commonly used in macroeconomics, may affect statistical inference. According with our analysis, we showed that, in the best case (this is, working with nonstationary data), the aggregation method does not have any incidence over the analysis. In the worst case (this is, working with stationary data), the aggregation affects the asymptotic distribution of the statistic associated with the estimated parameters, and may result in nonsense inference, this is the null hypothesis would be over rejected.

The present study should be understood as a warning call to empirical macroeconomists. Data aggregation and smoothing techniques may provoke statistical issues that can arise when drawing inference. From the point of view of an econometrician, aggregation should be employed with precaution.

We are aware, of course, that, from the macroeconomist perspective, aggregation aims to correct some problems in the data, such as cyclical fluctuations, in order to know the true nature of the data. In this case, we insist, the practitioner must be careful: When the variables behave as nonstationary processes, the risk of drawing spurious inference remains unaltered; this is, data aggregation and smoothing techniques do not reduce the risk of severe size distortions in standard testing procedures. When the variables are stationary, data aggregation and smoothing techniques actually increase the risk of drawing invalid inference. Under specific circumstances, a solution is possible, such as switching standard critical values with more appropriates ones. In many other cases, we can only suggest to the practitioner to take as many precautions as possible.

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Appendix A

Proof of Theorem 2.3.1

Let the *Data-Generating Process (DGP)* be the following:

$$y_t = u_{yt},$$

where $u_{yt} \sim iid(0, \sigma_y^2)$. This represents the *DGP* for the alternative hypothesis of the Dickey-Fuller test, i.e., stationary data. The econometrician can test her data running the following equation:

$$y_t = \alpha y_{t-1} + \varepsilon_t$$

If we substract y_{t-1} in each side of the equation, we obtain:

$$
\begin{aligned}
y_t - y_{t-1} &= \alpha y_{t-1} - y_{t-1} + \varepsilon_t \\
\Delta y_t &= (\alpha - 1) y_{t-1} + \varepsilon_t \\
\Delta y_t &= \gamma y_{t-1} + \varepsilon_t \\
\Delta u_t &= \gamma u_{t-1} + \varepsilon_t
\end{aligned}
$$

The t_γ statistic associated with γ takes the following form:

$$
t_\gamma = \frac{\hat{\gamma}}{\sqrt{\hat{\sigma}_\gamma^2}},
$$

where:

$$
\hat{\sigma}_\gamma^2 = \hat{\sigma}^2 (X'X)^{-1}
$$

We will use the asymptotic results (available in Hamilton (1994)):

$$
T^{-\frac{1}{2}} \sum u_{yt} u_{yt-1} \xrightarrow{D} \sigma_y^2 \omega_y(1)
$$

$$
T^{-1} \sum u_{yt}^2 \xrightarrow{P} \sigma_y^2
$$

Then, solving the expression for the t_γ statistic:

$$\begin{aligned}
\hat{\gamma} &= (X'X)^{-1}(X'Y) \\
&= \frac{\sum(u_{yt}-u_{yt-1})u_{yt-1}}{\sum u_{yt-1}^2} \\
&= \frac{\sum u_{yt}u_{yt-1} - \sum u_{yt-1}^2}{\sum u_{yt-1}^2} \\
&\xrightarrow{D} \frac{T^{\frac{1}{2}}\sigma_y^2\omega_y(1) - T\sigma_y^2}{T\sigma_y^2} \\
&\xrightarrow{D} T^{-\frac{1}{2}}\omega_y(1) - 1
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma}_y^2 &= \frac{\sum \varepsilon_t^2}{T} \\
&= \frac{1}{T}\left[\sum(\Delta u_{yt} - \hat{\gamma}u_{yt-1})^2\right] \\
&= \frac{1}{T}\left[\sum(u_{yt}-u_{yt-1})^2 - 2\hat{\gamma}\sum(u_{yt}-u_{yt-1})u_{yt-1} + \hat{\gamma}^2\sum u_{yt-1}^2\right] \\
&= \frac{1}{T}\left[\sum u_{yt}^2 + \sum u_{yt-1}^2 - 2\sum u_{yt}u_{yt-1} - 2\hat{\gamma}\sum u_{yt}u_{yt-1} + 2\hat{\gamma}\sum u_{yt-1}^2 + \hat{\gamma}^2\sum u_{yt-1}^2\right] \\
&\xrightarrow{D} \frac{1}{T}\left[T\sigma_y^2 + T\sigma_y^2 - 2T^{\frac{1}{2}}\sigma_y^2\omega_y(1) - 2\hat{\gamma}T^{\frac{1}{2}}\sigma_y^2\omega_y(1) + 2\hat{\gamma}T\sigma_y^2 + \hat{\gamma}^2T\sigma_y^2\right] \\
&\xrightarrow{D} \frac{\sigma_y^2(T-\omega_y(1))}{T}
\end{aligned}$$

$$
\begin{aligned}
(X'X)^{-1} &= \frac{1}{T\sum u_{yt}^2} \\
&\xrightarrow{P} \frac{1}{T\sigma_y^2}
\end{aligned}
$$

If we used only the first orders of convergence, we would have:

$$\hat{\gamma} \xrightarrow{P} -1$$

$$\hat{\sigma}^2 \xrightarrow{P} \sigma^2$$

If we substitute the expressions above, we would have the following t statistic:

$$
\begin{aligned}
t_\gamma &\xrightarrow{D} \frac{T^{-\frac{1}{2}}\omega_y(1) - 1}{\sqrt{\frac{1}{T\sigma_y^2}\frac{\sigma_y^2(T-\omega_y^2(1))}{T}}} \\
&\xrightarrow{D} \frac{\sqrt{T}\omega_y(1)}{\sqrt{T-\omega_y^2(1)}} - \frac{T}{\sqrt{T-\omega_y^2(1)}} \\
&\xrightarrow{D} \omega_y(1) - \sqrt{T}
\end{aligned}
$$

Appendix B

Proof of Theorem 3.3.3

Suppose the following nonstationary *Data-Generating Process*:

$$z_t \quad = \quad z_{t-1} + u_{zt},$$

where $z = x, y$ and $u_{zt} \sim iid(0, \sigma_z^2)$. Solving each equation, we would obtain:

$$z_t \quad = \quad \sum u_{zt},$$

Then, the practitioner decides apply the simple average method to her data. The following expression represent the transformation over her DGP. To simplify the process and without loss of generality, we will assume that the average is every three observations:

$$
\begin{aligned}
z_3^* &= \frac{z_1 + z_2 + z_3}{3}, \\
&= \frac{3u_{z,1} + 2u_{z,2} + u_{z,3}}{3}, \\
z_6^* &= \frac{z_4 + z_5 + z_6}{3}, \\
&= \frac{3u_{z,1} + 3u_{z,2} + 3u_{z,3} + 3u_{z,4} + 2u_{z,5} + u_{z,6}}{3}, \\
z_9^* &= \frac{z_7 + z_8 + z_9}{3}, \\
&= \frac{3u_{z,1} + 3u_{z,2} + 3u_{z,3} + 3u_{z,4} + 3u_{z,5} + 3u_{z,6} + 3u_{z,7} + 2u_{z,8} + u_{z,9}}{3}, \\
\vdots \quad & \quad \vdots \\
z_T^* &= \frac{z_{T-2} + z_{T-1} + z_T}{3}, \\
&= \frac{3u_{z,1} + 3u_{z,2} + 3u_{z,3} + 3u_{z,4} + \ldots + 3u_{z,T-2} + 2u_{z,T-1} + u_{z,T}}{3},
\end{aligned}
$$

which results on:

$$
\begin{aligned}
\sum y_t &= \frac{Tu_{z,1}+(T-1)u_{z,2}+\ldots+2u_{z,T-1}+u_{z,T}}{3}, \\
&= \sum_{t=0}^{T}\frac{(T-t)u_{z,t+1}}{3}, \\
&= \frac{(T+1)\sum u_{z,t}}{3}-\frac{\sum tu_{z,t}}{3}.
\end{aligned}
$$

According with Hamilton (1994) results, we know that:

$$
(T+1)\sum u_{z,t}-\sum tu_{z,t} \quad \xrightarrow{D} \quad T^{\frac{3}{2}}\sigma_z\omega_z(1)+T^{\frac{1}{2}}\sigma_z\omega_z(1)-T^{\frac{3}{2}}\sigma_z\omega_z(1)+T^{\frac{3}{2}}\sigma_z\int\omega_z(1)\delta r
$$

Then:

$$
T^{-\frac{3}{2}}\sum z_t^* \quad \xrightarrow{D} \quad \sigma_z\int\omega_z(1)\delta r
$$

The practitioner may run the following regression in order to test her data:

$$
y_t^* \quad = \quad \alpha+\beta x_t^*+\varepsilon_t
$$

Therefore, the vector of coefficients is:

$$\beta = (X'X)^{-1}(X'Y),$$

where:

$$x_t = \begin{bmatrix} 1 & x_1^* \\ 1 & x_2^* \\ 1 & x_3^* \\ \vdots & \vdots \\ 1 & x_T^* \end{bmatrix}$$

To simplify the notation, z_t^* will be written as z_t for $z = x, y$. Then:

$$(X'X) = \begin{bmatrix} T & \sum x_t \\ \sum x_t^2 & \sum x_t^2 \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{T\sum x_t^2 - [\sum x_t]^2} \begin{bmatrix} \sum x_t^2 & -\sum x_t \\ -\sum x_t & T \end{bmatrix}$$

$$
(X'Y) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix}
$$

$$
= \begin{bmatrix} \sum y_t \\ \sum x_t y_t \end{bmatrix}
$$

Then:

$$
(X'X)^{-1}(X'Y) = \frac{1}{T\sum x_t^2 - [\sum x_t]^2} \begin{bmatrix} \sum x_t^2 & -\sum x_t \\ -\sum x_t & T \end{bmatrix} \begin{bmatrix} \sum y_t \\ \sum x_t y_t \end{bmatrix}
$$

Then, the expression for each coefficient would be:

$$
\hat{\alpha} = \frac{\sum x_t^2 \sum y_t - \sum x_t \sum x_t y_t}{T\sum x_t^2 - [\sum x_t]^2}
$$

$$
\hat{\beta} = \frac{T\sum x_t y_t - \sum x_t \sum y_t}{T\sum x_t^2 - [\sum x_t]^2}
$$

Then, using the expressions from Hamilton (1994), we obtain:

$$T^{-\frac{1}{2}}\hat{\alpha} \quad \xrightarrow{D} \quad \frac{\sigma_y \left(\int \omega_x(r) \int \omega_x(r)\omega_y(r) - \int \omega_x(r)^2 \int \omega_y(r)\right)}{(\int \omega_x(r))^2 - \int \omega_x(r)^2},$$

$$\hat{\beta} \quad \xrightarrow{D} \quad \frac{\sigma_y(\int \omega_x(r) \int \omega_y(r) - \int \omega_x(r)\omega_y(r))}{(\int \omega_x(r))^2 - \int \omega_x(r)^2},$$

The t-ratios for α and β and the expressions for the R-squared and $\mathcal{F}$ statistic are obtained in a similar way.

Appendix C

Gamma expressions in Theorems 3.3.1 - 3.3.5

Here are the expressions for the Γ's used in section 3.3

$$\Gamma_1 = 2\int \omega_x(r)\int \omega_y(r)\int \omega_x(r)\omega_y(r)$$

$$\Gamma_2 = \int \omega_x(r)^2\left(\int \omega_y(r)\right)^2 + \int \omega_y(r)^2\left(\int \omega_x(r)\right)^2 - \int \omega_y(r)^2\int \omega_x(r)^2$$

$$\Gamma_3 = \left(\left(\int \omega_x(r)\right)^2 - \int \omega_x(r)^2\right)\left(\left(\int \omega_x(r)\right)^2 - \int \omega_x(r)^2\right)$$

$$\Gamma_4 = -4(\mu_y^2)(\int \omega_x(r)^2) + (\mu_y^2)\int \omega_x(r)^2 + 12(\mu_y^2)\int \omega_x(r)\int r\omega_x(r) - 3\mu_x\int r\omega_y(r)$$

$$\Gamma_5 = 12(\mu_y^2)(\int r\omega_x(r)^2) + 8\mu_y\mu_x\int \omega_x(r)\int \omega_y(r) - 12\mu_x\mu_y\int r\omega_x(r)\int \omega_y(r)$$

$$
\begin{aligned}
\Gamma_6 &= 24\mu_x\mu_y \int r\omega_x(r) \int r\omega_y(r) + 12(\mu_x^2) \int \omega_y(r) \int r\omega_y(r) - 12(\mu_x^2)(\int r\omega_y(r)^2) \\
&\quad -2\mu_y\mu_x \int \omega_x\omega_y(r)
\end{aligned}
$$

$$
\Gamma_7 = 3\mu_x \int r\omega_y(r) - 4(\mu_x^2)(\int \omega_y(r)^2) + (\mu_x^2) \int \omega_y(r)^2
$$

$$
\Gamma_8 = 12\mu_x\mu_y \int \omega_x(r) \int r\omega_y(r)
$$

Appendix D

Distributions for aggregated DGPs with unit root

We show that the asymptotic distribution of a DGP with unit root (i.e., a DGP with form 3.3 or 3.4) has no distortion after being aggregated. To prove it, we plotted both the asymptotic distribution of the t-ratio associated with the estimated parameter of DGPs 3.3 and 3.4 and the asymptotic distribution obtained through a Monte-Carlo experiment. Figures D.1 and D.2 present our results.

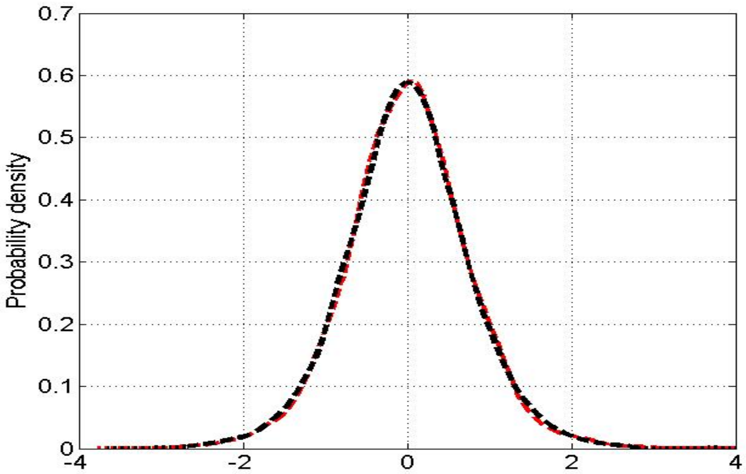

Figure D.1: **Distribution of the t-ratio for the DGP with unit root (represented by 3.3) and is aggregated by Moving or Simple Average.** The black dashed line represents the probability density of the DGP with form 3.3 without any aggregation procedure. The red dashed line shows the probability density for the t-ratio associated with the estimated parameters of the DGP 3.3 which has been aggregated by 3.7 or 3.8 (any aggregation procedure results in the same asymptotic distribution) for $T = 1,000$ and 10,000 replications. Since the distribution are the same, there is no effect of the aggregation methods on the asymptotic distribution of the DGP with unit root.

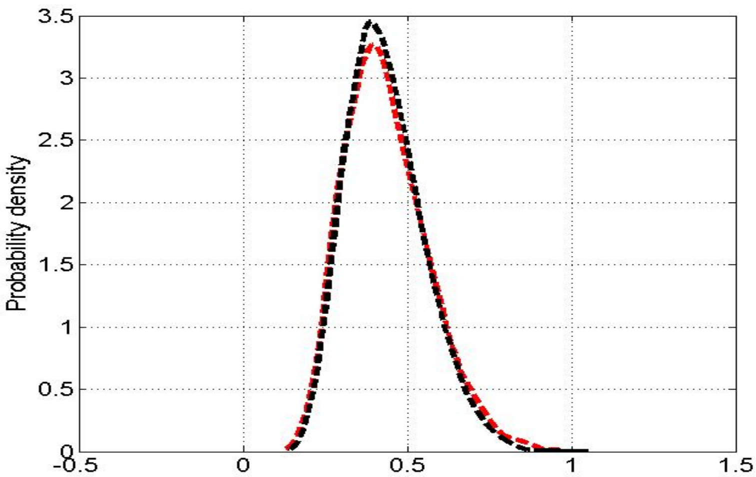

Figure D.2: **Distribution of the t-ratio for the DGP with unit root, a drift (represented by 3.4) and is aggregated by Moving or Simple Average.** The black dashed line represents the probability density of the DGP with form 3.4 without any aggregation procedure. The red dashed line shows the probability density for the t-ratio associated with the estimated parameters of the DGP 3.4 which has been aggregated by 3.7 or 3.8 (any aggregation procedure results in the same asymptotic distribution) for $\mu_x = 0.7$, $\mu_y = 0.4$, $T = 1,000$ and 10,000 replications. Since the distribution are the same, there is no effect of the aggregation methods on the asymptotic distribution of the DGP with unit root.

Appendix E

Matlab code for Theorem 3.3.3

```
% Topics in Macroeconometrics-Inference under aggregation
% Guillermo Verduzco and Daniel Ventosa
% July, 2016

% Code for theorem 3.3.3

clear all

% Number of replications
iter=10000;
T=1000;


% The aggregation method will be applied with the smoothMM
    function developed by Daniel Ventosa-Santaularia. The
    information is:

```

```
% This function smooths the series using simple moving
    averages or averages
% Input:
%    1.- X:       The T x 1 series to be smoothed
%    2.- k:       The span of the moving average
%    3.- M:       Choose whether you want Moving averages or
    simple averages
%                      1. Moving average
%                      2. Simple average
% Output:
%    1.- Xmm:     The (T-k) x 1 or floor(T/k) x 1 smoothed
    series
%_______________________________________________________
for j=1:iter

% This section prepares the matrix of the data
K=1;
ux=randn(T,1);
uy=randn(T,1);
Xm=zeros(T,1);
Ym=zeros(T,1);
X_p=zeros(T,1);
Y_p=zeros(T,1);
X_as=zeros(T,1);
Y_as=zeros(T,1);
X_as(1,1)=ux(1);
Y_as(1,1)=uy(1);

```

```
%_____________________________________________________
% Data-generating process with unit root.
Xm=cumsum(randn(T,1));
Ym=cumsum(randn(T,1));

%_____________________________________________________
% This generates the aggregated data
R=3;
M=1;

X_p=smoothMM(Xm,R,M);
Y_p=smoothMM(Ym,R,M);

S=length(X_p);

%_____________________________________________________
% Monte-Carlo results:
Xb=[ones(S,1),X_p];
Results=ols(Y_p,Xb);
Alpha=Results.beta(1,1);
Alpha_hat(j,:)=Alpha/sqrt(T);
Beta=Results.beta(2,1);
Beta_hat(j,:)=Beta;
Varianza = sum((Results.resid).^2)/T^2;
Talpha_g=Results.tstat(1,1);
Talpha_hat(j,:)=Talpha_g/sqrt(T);
Tbeta_g=Results.tstat(2,1);
Tbeta_hat(j,:)=Tbeta_g/sqrt(T);
```

```
Rsquared_g=Results.rsqr;
Rsquared_hat(j,:)=Rsquared_g;
F_hat(j,:) = T*Rsquared_g/(1-Rsquared_g);

%______________________________________________________

% Asymptotic results using the Brownian motion

Xp=cumsum(randn(T,1));
Yp=cumsum(randn(T,1));

Xs=smoothMM(Xp,R,M);
Ys=smoothMM(Yp,R,M);

Sxs=sum(Xs)/(T^(3/2));
Sys=sum(Ys)/(T^(3/2));
Sxys=sum(Xs.*Ys)/((T^2));
Sxs2=sum(Xs.^2)/((T^2));
Sys2=sum(Ys.^2)/((T^2));

Alpha_asymp(j,:) = (Sxs*Sxys-Sxs2*Sys)/((Sxs^2)-Sxs2);
Beta_asymp(j,:) = (Sxs*Sys-Sxys)/((Sxs^2)-Sxs2);
Sigma = (Sxys^2-2*Sxs*Sxys*Sys+Sxs2*Sys^2+Sxs^2*Sys2-Sxs2*Sys2)
    /(Sxs^2-Sxs2);
Talpha_asymp(j,:) = (Sxs*Sxys-Sxs2*Sys)/((Sxys^2-2*Sxs*Sxys*Sys
    +Sxs2*Sys^2+Sxs^2*Sys2-Sxs2*Sys2)*-Sxs2)^(1/2);
Tbeta_asymp(j,:) = (Sxs*Sys-Sxys)/((Sxys^2-2*Sxs*Sxys*Sys+Sxs2*
    Sys^2+Sxs^2*Sys2-Sxs2*Sys2)*-1)^(1/2);
```

```
Rsquared_asymp(j,:) = 1+((Sxys^2-2*Sxs*Sxys*Sys+Sxs2*Sys^2+Sxs^2*Sys2-Sxs2*Sys2)/(((Sxs^2)-Sxs2)*((Sys^2)-Sys2)));
F_asymp(j,:)= -T*((((Sxs^2)-Sxs2)*((Sys^2)-Sys2))+(Sxys^2-2*Sxs*Sxys*Sys+Sxs2*Sys^2+Sxs^2*Sys2-Sxs2*Sys2))/(Sxys^2-2*Sxs*Sxys*Sys+Sxs2*Sys^2+Sxs^2*Sys2-Sxs2*Sys2);
end

%_____________________________________________________________
% Densities to be plotted
[Fas,as]=ksdensity(Alpha_hat);
[Faa,aa]=ksdensity(Alpha_asymp);

[Fbs,bs]=ksdensity(Beta_hat);
[Fba,ba]=ksdensity(Beta_asymp);

[Fvar,tvar]=ksdensity(Varianza);
[Fsigma,tsigma]=ksdensity(Sigma);

[Ftas,tas]=ksdensity(Talpha_hat);
[Ftaa,taa]=ksdensity(Talpha_asymp);

[Ftbs,tbs]=ksdensity(Tbeta_hat);
[Ftba,tba]=ksdensity(Tbeta_asymp);

[Frs,rs]=ksdensity(Rsquared_hat);
[Fra,ra]=ksdensity(Rsquared_asymp);

[Ffs,fs]=ksdensity(F_hat);
```

```
[Ffa,fa]=ksdensity(F_asymp);

%_________________________________________________________
% This create a standard normal distribution
x = [-5:.1:5];
norm = normpdf(x,0,1);

%_________________________________________________________
% Figures of the densities
figure(1)
subplot(2,2,1)
plot(as,Fas,'r',aa,Faa,'k')
title('\alpha_{e}, \alpha_{asymp}');
subplot(2,2,2)
plot(bs,Fbs,'r',ba,Fba,'k')
title('\beta_{e}, \beta_{asymp}');
subplot(2,2,3)
plot(tas,Ftas,'r',taa,Ftaa,'k',x,norm,'b')
title('t_{\alpha_{e}}, t_{\alpha_{asymp}}');
subplot(2,2,4)
plot(tbs,Ftbs,'r',tba,Ftba,'k',x,norm,'b')
title('t_{\beta_{e}}, t_{\beta_{asymp}}');

figure(2)
subplot(1,1,1)
plot(tvar,Fvar,'r',tsigma,Fsigma,'k')
title('Varianza');

```

```
figure(3)
subplot(2,1,1)
plot(rs,Frs,'r',ra,Fra,'k')
title('R^2');
subplot(2,1,2)
plot(fs,Ffs,'r',fa,Ffa,'k')
title('F');

figure(4)
plot(tba,Ftba,'k',x,norm,'b')
title('t_{\beta_{e}}, t_{\beta_{asymp}}');

%_______________________________________________
```