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THE KUPERBERG CONJECTURE IS TRUE IN TWO DIMENSIONS: ANY STRICTLY COMPACT C^2 FIGURE CAN BE FIXED FIRMLY BY THREE POINTS UNLESS IT IS THE DISK

Abstract. We treat the immobilization problem introduced by Kuperberg and Papadimitriou in the case of two dimensional strictly convex figures, proving a strong version of the Kuperberg conjecture: any C^2 strictly convex figure (supporting tangent lines touch at one point) may be fixed by three points satisfying the second order condition unless it is the disk.

Introduction

Immobilization problems where introduced by W. Kuperberg [K] and Papadimitriou [MNP1]. They were motivated by grasping problems in robotics [MNP1, 2]. Interest then developed in the purely geometrical aspect of the problem. Focusing on smooth convex curves, in [BMU], geometrical conditions were obtained for the first and second order conditions of immobilization for plane figures, and it was proved that analytic convex figures other than the disk may be fixed by three points. In [BFMM], focusing instead on tetrahedra, the first order necessary condition was shown to imply the second order sufficient condition in the three dimensional case.

In a separate article we have treated the n-dimensional case, writing out the zeroeth, first and second order conditions and obtaining some general theorems(see [M]). Here we prove Kuperberg's conjecture in the two dimensional case for twice differentiable convex curves.

The main ideas are the following. Let us first outline an interpretation of the meaning of the conditions determining whether a C^2 two dimensional figure K is fixed by three points p_0, p_1, p_2 chosen on its boundary. We examine if it is possible to wiggle the figure free from these points. The zeroeth order condition states that just a translation will not separate the figure from these points. Under this condition, we may suppose that wiggling is the result of rotating the figure and then translating it so that two of the points, say p_0, p_1, have paths along the boundary of the figure. The question is then whether if this is done the third point, p_2, follows a path remaining on the outside or penetrating the interior of the figure. The first order condition states that the third point follows a path initially tangent to the boundary of the figure. It is equivalent to the concurrence at some point q of the lines L_i through p_i each in the direction of the normal N_i to ∂K at p_i. The second order condition states that the third point locally penetrates the interior of the figure.

Let us outline the method we shall be using. Analytically, the problem is equivalent to finding triples p_0, p_1, p_2, which are solutions of an equation (the first order condition) satisfying certain inequality constraints (the zeroeth and second order conditions). We first find triples close to a solution by considering the largest disk C inscribed by the figure, which generically yields three points fixing K. To solve the non-generic cases we then choose a point of concurrence q close to the center of C, along a ray in a direction which will guarantee the existence of triples satisfying the zeroeth order condition. The solution set p_i of the first order condition in terms of the choice of q along its ray turns out to be equivalent to the level sets of a C^1 function ϕ (defined along ∂K) in terms of the level α. The sign of the derivative of ϕ then gives information on the sign of the contribution of each point to the second order condition (itself a continuous function not amenable to differentiable treatment). Thus Sard's theorem yields levels α for which the solutions p of $\phi(p) = \alpha$ will have non-zero derivatives. We combine this with the topological fact that if ϕ is known to increase (decrease) between the endpoints of an interval (this will be implied by the geometry and the assumption of strict convexity) then, in between, some of the solutions p must have positive (negative) derivatives. Thus we find points p_i satisfying the first order condition and contributing appropriately to the second order condition. In some of the cases an additional complication arises, when the contribution to the second order condition of some point must compensate for the negative contribution of some other point, for which a more delicate estimate of the derivative of ϕ must be obtained. Finally, the calculations are much simplified by the choice of functions used to describe closed strictly compact curves, defined in the appendix.

2. *Definitions*

Let us make precise the concepts of "immobilization" and "trapping". We follow the notation found in [BFMM]. Let $\mathscr{E}$ be the Lie Group of orientation preserving isometries of Euclidean space $\mathbb{R}^2$. Given any two sets X, $Y \subseteq \mathbb{R}^2$, define the motions of X in Y to be

$$\mathscr{E}(X, Y) = \{g \in \mathscr{E} \mid g(X) \subset Y\}.$$

Throughout this article, let $K \subset \mathbb{R}^2$ be a compact figure with non-empty interior. Denote by Int$K$ the interior of K, and by $\mathbb{C}K$ its "outside", that is, $\mathbb{C}K = \mathbb{R}^2 \setminus \text{Int}K$, so that $K \cap \mathbb{C}K = \partial K$.

2.1. Definition. We say that P *immobilizes* (*or fixes*) K if $P \subset \mathbb{C}K$ and the identity map id $\in \mathscr{E}$ is an isolated point component of $\mathscr{E}(P, \mathbb{C}K)$ (with respect to path connectedness). We say that P *traps* K if $P \subset \mathbb{C}K$ and the connected component of id $\in \mathscr{E}$ is compact.■

The exceptional cases to immobilization (such as those posed by spheres or screws) are cases in which points which almost fix a figure can slide along its surface. In these cases we say K admits a thread.

2.2. Definition. We say that K admits a *global (local) thread* (on its surface) if there exists a set $P \subset \mathbb{C}K$ which traps K and which satisfies the property: for every $g \in \mathscr{E}(P, \mathbb{C}K)$ in the connected component of id $\in \mathscr{E}$ (or only in a neigbourhood of id), $g(P) \cap \partial K \neq \phi$. We say that P generates a thread.■

It is clear that each $g(P)$ traps K. The idea is that the union of sets $g(P) \cap \partial K$ is what in simple cases such as the surface of a screw we call a thread.

3. The Zeroeth, First and Second Order Conditions

We quote conditions under which a set of 3 points $P = \{p_0, p_1, p_2\}$ fixes a 2-dimensional figure K at differentiable points on the boundary (see [M]). Let the set of outward normals corresponding to these points be $N = \{N_0, N_1, N_2\}$.

3.1. Proposition. A necessary and sufficient condition for the points P to fix a C^1 figure K up to translations (the *zeroeth order* condition) is that any two vectors of N be linearly independent and that there exist positive constants $a_0, a_1, a_2 > 0$ for which the set N of normals satisfies $\Sigma_0^2 a_i N_i = 0$.■

For points p_0, p_1, p_2 satisfying the zeroeth order condition we write $n_i = a_i N_i$; $\Sigma_0^n n_i = 0$. The $a_i > 0$ are defined up to a constant. For definiteness suppose p_i are arranged counterclockwise on ∂K. The numbers a_i may then be chosen as follows:

$$a_i = |N_k N_j| = \begin{vmatrix} \cos\theta_k & \cos\theta_j \\ \sin\theta_k & \sin\theta_j \end{vmatrix} = \sin(\theta_k - \theta_j) > 0, \qquad (3.1.2)$$

where (i, j, k) is the even permutation of $\{1, 2, 3\}$ written with an i at the beginning.

3.2. Proposition. Suppose the points p_0, p_1, p_2 fix a C^2 figure K up to translations, and $\Sigma_0^n n_i^T p_i \neq 0$ on ∂K (this is true for starshaped figures K). The necessary first order condition for p_i to fix K is

$$\Sigma_0^2 p_i \cdot n_i = 0 \, . \qquad (3.2.1)$$

The sufficient second order condition for p_i to fix K is

$$\Sigma_0^2 a_i \left(N_i \cdot p_i - k_i (N_i \cdot p_i)^2 \right) > 0, \qquad (3.2.2)$$

where k_i is the curvature of ∂K at p_i.■

3.3. Definition. We say that a C^2 figure K held at 3 points is fixed *firmly* if it satisfies the zeroeth, first, and second order conditions.■

In the two dimensional case the first order condition has the following characterization.

3.4. Theorem. Let $p_i, n_i, 0 \le i \le 2$ be a set of 3 points and directions defining lines L_i. Then $\Sigma_0^2 p_i \cdot n_i = 0$ if and only if L_i are concurrent.■

4. Fixing Strictly Convex Bodies: The 2-dimensional Case

Let K be a C^2 strictly convex figure (supporting tangent lines touch at one point). Therefore the curvature exists and is non-negative. We thus allow the curvature to become zero at some points of ∂K. r is defined almost everywhere on the Gauss map image, and is integrable along any path.

We begin by quoting a theorem about the largest inscribed balls in a two dimensional figure.

4.1. Theorem. *a)* Suppose a closed disk contained in a C^1 figure $K \subset \mathbb{R}^2$ touches it only on an open semi-circumference. Then there is a bigger disk contained in K.

b) Suppose an inscribed closed disk in a C^1 figure K contains on its intersection with ∂K a set of 3 points $P = p_0, p_1, p_2$ whose corresponding normals N_i satisfy the zeroeth order condition. Then either P fixes K or it generates a thread.■

Let us now prove a simple analytical proposition which we shall use below.

4.2. Proposition. *1)* Let $\phi : [a, b] \to \mathbb{R}$ be continuous on the closed and C^1 and positive on the open interval, and suppose $\phi(a) = 0$. Let $\phi_{\max} = \sup_{[a,b]} \phi > 0$. Then a.a. ("for almost all") $\alpha \in (0, \phi_{\max})$ there exists $p \in (a, b)$ satisfying $\phi(p) = \alpha$ and $\phi'(p) > 0$.

2) Let $\phi : [a, b] \to \mathbb{R}$ be continuous on the closed and C^1 and positive on the open interval, with $\phi(a) = 0$.

$$\forall\, c > 0, \beta \in (a, b]\ \exists\ \text{ open } S \subset [a, \beta] :\ \phi' > c\,\phi \text{ on } S.$$

Hence $\forall\ \alpha_1 \in (0, \phi_{\max})\ \exists$ open $A \subseteq (0, \alpha_1)$:

$$\forall\, \alpha \in A\ \exists\, p \in (a, b) :\ \phi(p) = \alpha \text{ and } \phi'(p) > c\alpha.$$

Proof. *1)* By Sard's theorem, a.a. $\alpha \in (0, \beta)$ every point $p \in (a, b)$ satisfying $\phi(p) = \alpha$ has $\phi'(p) \neq 0$. Clearly at least one of these has a positive derivative, since otherwise ϕ could not rise to β.

2) Suppose, on the contrary that there exist $c > 0$, $\beta \in (a, b]$ for which $\phi' \le c\,\phi$ on $(a, \beta]$.Then $(ln\phi)' \le c$ on (a, β), so that $[ln\phi]_a^\beta \le c(\beta - a)$. But therefore $\ln\phi(x)$ cannot tend to $-\infty$ as $x \to a$, which contradicts $\phi(0) = 0$.

Now choose any $\alpha_1 \in (0, \phi_{\max})$, and β small enough so $\phi(0, \beta) \subseteq (0, \alpha)$. By what we just showed, there is an open set $S \subset [a, \beta]$ on which $\phi' > c\phi$. Let $A = \phi(S)$.■

We now aim at proving a lemma which will allow us to find points to fix convex bodies. The calculations have been greatly simplified by using the functions defined in the appendix, upon which this section is completely dependent (and which must be read first), but which has been kept separate since it belongs to a different topic.

Define for any point with Gaus coordinate θ the line $L(\theta)$ to be the one going through $f(\theta)$ along the direction of the normal N. Since $n = 2$, the first order condition implies that the lines $L_i = L(\theta_i)$ through p_i concur at some point $q = \alpha(\cos\xi, \sin\xi)^T$. We shall vary α. Define the functions g^α, h^α by fixing the origin at q, that is, by requiring

$$f - q = g^\alpha N + h^\alpha T. \tag{4.3.1}$$

Then

$$g^\alpha = g - \alpha\cos(\theta - \xi),\ h^\alpha = h + \alpha\sin(\theta - \xi). \tag{4.3.2}$$

It now follows that $h_i^\alpha = h^\alpha(\theta_i) = 0$, $i = 0, 1, 2$. Thus, finding points satisfying the first order condition is equivalent to finding an origin q and values of θ_i for which $h_i^\alpha = 0$. Let us now write the second order condition in these terms. The second order condition takes the form

$$\Sigma_0^2\, a_i\left(g_i^\alpha - k_i\,(g_i^\alpha)^2\right) = \Sigma_0^2\, a_i g_i^\alpha \frac{\mathrm{d}}{\mathrm{d}s}(h_i^\alpha) > 0 \tag{4.3.3}$$

since $(An_i)\cdot(Ap_i) = n_i^T p_i = g_i$ and $B_i(p_i', p_i') = k_i\, g_i^2$.

Now define the function

$$\phi(\theta) = -\frac{h(\theta)}{\sin(\theta - \xi)}. \tag{4.3.4}$$

Suppose θ^α solves $\phi(\theta^\alpha) = \alpha$. Then $h^\alpha(\theta^\alpha) = 0$, while

$$\phi'(\theta^\alpha) = -\frac{h'(\theta^\alpha)}{\sin(\theta^\alpha - \xi)} + \frac{h(\theta^\alpha)}{\sin^2(\theta^\alpha - \xi)}\cos(\theta^\alpha - \xi) = -\frac{h^{\alpha\prime}(\theta^\alpha)}{\sin(\theta^\alpha - \xi)}.$$

Thus the contribution of the point $f(\theta^\alpha)$ to the second order condition is its a_i (which depends on the remaining points) multiplied by $-(g^\alpha k\phi' \sin(\theta - \xi))\,|_{\theta = \theta^\alpha}$. We may now state our lemma.

By statements such as "$|f| > 1$ on a set I arbitrarily close to $\theta \in \bar{I}$" we shall mean that $\forall\, \varepsilon > 0\ \exists\, \chi \in I : |\chi - \theta| < \varepsilon$ and $|f(\chi)| > 1$.

4.3. Lemma. *a)* Suppose we are given a point $p \in K$ with Gauss coordinate ω, for which $|f(\omega)| = 1$, and for which there is a neighbourhood $(\omega, \omega + \varepsilon)$ on which $|f| > 1$ arbitrarily close to ω. Let $\xi \in (\omega, \omega + \pi)$. *b)* Alternatively, $|f| > 1$ on $(\omega - \varepsilon, \omega)$ arbitrarily close to ω, and $\xi \in (\omega - \pi, \omega)$.

Instead, suppose $|f(\omega)| = 1$, $h(\omega) = h_\theta(\omega) = 0$, and $|f| > 1$ on (c) $(\omega, \omega + \varepsilon)$ or (d) $(\omega - \varepsilon, \omega)$, arbitrarily close to ω. Let $\xi = \omega + \pi$.

Define ϕ by 4.3.4. We may conclude that:

1) $\exists\, \alpha_0 > 0$ for which a.a. $\alpha \in (0, \alpha_0)$ there is a point $\theta^* \in (\omega, \omega + \varepsilon)$ (hypothesis a, c) or $\theta^* \in (\omega - \varepsilon, \omega)$ (b, d) satisfying $\phi(\theta^*) = \alpha$ at which $\phi_s > 0$ (hypothesis a, c) or $\phi_s < 0$ (b, d).

2) Given any $c > 0$, there exists $\alpha_0 > 0$ such that $\forall\ \alpha_1 \in (0, \alpha_0)\ \exists$ open $S \subset (0, \alpha_1):\ \forall\, \alpha \in S$

$$\exists \begin{cases} \theta^* \in (\omega, \omega + \varepsilon) & (\text{a, c}) \\ \theta^* \in (\omega - \varepsilon, \omega) & (\text{b, d}) \end{cases} : \phi(\theta^*) = \alpha^*, \begin{cases} \phi_s(\theta^*) > c\alpha^* & (\text{a, c}) \\ \phi_s(\theta^*) < -c\alpha^* & (\text{b, d}) \end{cases}$$

(the parenthesis indicate the cases to which the statements apply).

Proof. *a)* Because h is the gradient of $\frac{1}{2}|f|^2$, there must be some value $\theta_1 \in (\omega, \min(\omega + \varepsilon, \xi))$ for which $h > 0$. Therefore, using the function ϕ defined in 4.3.4, and the hypothesis, which imply $h(\omega) = 0$, we have $\phi(\omega) = 0$, $\alpha_1 = \phi(\theta_1) > 0$. Applying proposition 4.2(1) on this to ϕ considered as a function of s, we get conclusion (1). To prove (2), let χ be the supremum of those numbers ζ for which ϕ is positive on $(\zeta, \theta_1]$ and $\phi(\zeta) = 0$. Apply proposition 4.2(2) to ϕ on $[\chi, \theta_1]$, considered as a function of s. There is an open set $S \subset [\chi, \theta_1] : \phi_s > c\phi$ on S. In the alternative case (b) the sign of the derivatives of ϕ reverses because h must be negative for $|f|$ to increase value as θ decreases.

For cases (c), (d) satisfying the hypothesis $h(\omega) = h_\theta(\omega) = 0$, $\xi = \omega + \pi$, the function ϕ may be regarded as continuous at $\theta = \omega$. This is because by L'Hôpital's rule

$$\lim_{\theta \to \omega} \phi(\theta) = \lim_{\theta \to \omega} \frac{h'(\theta)}{\cos(\theta - \omega)} = 0.$$

The rest of the argument is similar.■

Observe for the applications that $h(\theta^*) = -\sin(\theta^* - \xi)\,\phi'(\theta^*) > 0$ in every case while the derivative $\phi_s(\theta^*)$ has been chosen so $h_s(\theta^*) > 0$. Also

$$g^{\alpha^*}(\theta^*) = g(\theta^*) - \alpha^* \cos(\theta^* - \xi) \to 1 \text{ as } \alpha^* \to 0.$$

Thus θ^* is the Gauss coordinate of a point which has $L(\theta^*)$ passing through $q = \alpha^*(\cos\xi, \sin\xi)^T$ and makes a positive contribution to the second order condition, for which we have in addition an estimate which we shall use for certain delicate cases to be encountered.

We are now ready to prove the theorem:

4.4. Theorem. Any strictly compact 2 dimensional C^2 figure K not admitting local threads may be fixed by three points.

Proof. Because ∂K is strictly compact, we are admitting the possibility that r be undefined. However, the Gauss map has a continuous inverse. Hence the function $s(\theta)$ exists and is absolutely continuous on compact sets, so its derivative $r = k^{-1}$ exists almost everywhere and is integrable. (For the differentiability almost everywhere of absolutely continuous functions see [A, §3.36].) Thus equations A.7 and A.10 hold wherever r is defined.

Let ρ be the supremum of the radii of circles strictly contained in K. By the compactness of K there exists some inscribed circle C with this radius.

Consider the set $I = C \cap \partial K$. Let $\mathcal{P}$ be the proposition: "$\exists\, Q = \{q_1, q_2, q_3\} \subseteq I$ s.t. $N(Q)$ satisfy the zeroeth order condition". Theorem 4.1 shows that $\mathcal{P}$ implies Q fixes K.

The rest of the proof is dedicated to the case in which $\mathcal{P}$ is false: $\forall\, Q = \{q_1, q_2, q_3\} \subseteq I$, $N(Q)$ lies on a (closed) semi-circumference (throughout the remaining discussion semi-circumferences are assumed closed unless explicitly stated otherwise). We first show that there must be two points on I with antipodal normals.

Let ϕ be the Gauss coordinate on ∂K and S^1. Take any two points of I with normals $\theta_1, \theta_2, \in S^1$, which are not antipodal. Suppose without loss of generality that $\theta_2 \in (\theta_1, \theta_1 + \pi)$, rather than $\theta_2 \in (\theta_1, \theta_1 - \pi)$ (where θ is an element of a set of angles if in that set there is an angle which differs from θ by an integer multiple of 2π). Then I must be contained in $[\theta_2 - \pi, \theta_1 + \pi]$, because any point in the complement, together with θ_1, θ_2, forms a set of three points whose normals do not lie on a semi-circumference. Since I is closed we have in I

$$\theta_I = \inf\, [\theta_2 - \pi, \theta_1 + \pi] \cap I,\ \theta_S = \sup[\theta_2 - \pi, \theta_1 + \pi] \cap I$$

and $I \subseteq [\theta_I, \theta_S]$. These cannot cover more than a semi-circumference because then θ_I, θ_S and one of θ_1, θ_2 would form a set of three points satisfying the zeroeth order condition. Neither can they cover less than the semi-circumference, since if all of I lies on an open semi-circumference, then C is note maximal, by 4.1(a). Therefore θ_I and θ_S are antipodal.

Hence by considering a largest inscribed circle C either three points exist on $C \cap \partial K$ which fix K or there are two points q_1, q_2 on $C \cap \partial K$ with antipodal normals. These must therefore lie on a diameter of C. It is further easy to see by cases that if $\mathcal{P}$ is false then either I consist of 4 points with antipodal normals by pairs, or I lies wholly on one semi-circumference.

Choose now the x axis to be the line joining q_1, q_2, with origin at the midpoint of these points. Introduce the functions f, N, T, g, h, θ defined in the appendix. Then

q_1, q_2 have coordinates $\theta = 0, \pi$, respectively. Simplify further by rescaling so that ρ, the radius of the largest inscribed circle, is 1.

In the case in which I lies wholly on one semi-circumference, let this correspond to $[\pi, 2\pi]$. It follows that in all cases there are neighbourhoods $(0, \omega_1)$, (ω_2, π) on which $|f| > 1$, and either both intervals are $(0, \pi)$ or $\omega_1 = \omega_2$.

The proof will be completed by showing that there exist three points p_0, p_1, p_2, on ∂K which satisfy the first and second order conditions (therefore fixing K firmly). We now consider several cases of the radius of curvature near q_1, q_2. The first is

$$r(0) > 1 \qquad \text{(Case 1)}$$

(the radius of curvature is greater than or equal to 1 anyway since otherwise ∂K would intersect the interior of the inscribed circle). The second is

$$r(0) = 1 \text{ and } |f| > 1 \text{ arbitrarily close to 0 for } \theta < 0. \qquad \text{(Case 2)}$$

The third,

$$r(0) = 1, |f(\theta)| \equiv 1 \text{ for } \theta \in (-\delta, 0] \qquad \text{(Case 3).}$$

Case 1. $r(0) > 1$. In this case we shall fix K with three points chosen as follows: p_1, p_2 slightly above q_1 and q_2, and p_0 with normal in the lower open semi-circumference (these points are disposed counterclockwise so $a_1, a_2, a_3 > 0$).

Observe that $h(0) = h(\pi) = 0$, $g(0) = g(\pi) = 1$ and $\frac{dg}{d\theta} = h$. By the, mean value theorem there exists $\zeta \in (\pi, 2\pi)$ with $h((\zeta) = 0$. Let ζ define p_0, set $\xi = \zeta - \pi$ and consider points of concurrence on the ray

$$\{q = \alpha(\cos\xi > \sin\xi)^T : \alpha > 0\}.$$

Write $Q(\theta_1, \theta_2) = \Sigma_0^2 a_i (g_i - k_i g_i^2)$ for the second order condition at the points given by $\zeta, \theta_1, \theta_2$ placed approximately as stated. Q is continuous in θ and, since $a_0 = \sin(\theta_2 - \theta_1)$, $a_1 = \sin(\zeta - \theta_2)$, $a_2 = \sin(\theta_1 - \zeta)$, $k(0) < 1$, $k(1) \leq 1$,

$$Q(0, 0) = \sin(\xi)\left(2 - k(0) - k(1)\right) > 0.$$

Choose ε so that $|f| > 1$ on the intervals $(0, \varepsilon)$, $(\pi - \varepsilon, \pi)$, and $Q(\theta_1, \theta_2) > 0$ on $(0, \varepsilon) \times (\pi - \varepsilon, \pi)$. Now apply lemma 4.3a(1) to the points p_1, p_2, obtaining positive values α^1, α^2 such that

$$\text{a.a.} \quad \alpha^* \in (0, \alpha^1)\ \exists\ \theta_1^* \in (0, \varepsilon) : \phi(\theta_1^*) = a^*, \phi_s(\theta_1^*) > 0;$$
$$\text{a.a.} \quad \alpha^* \in (0, \alpha^2)\ \exists\ \theta_2 \in (\pi - \varepsilon, \pi) : \phi(\theta_2^*) = \alpha^*, \phi_s(\theta_2^*) < 0.$$

It is enough to see that by construction for any $\alpha^* \in (0, \alpha^1) \cap (0, \alpha^2)$ there exist lines $L(\theta_1^*), L(\theta_2^*), L(\zeta)$ concurring at $q = \alpha^*(\cos\xi, \sin\xi)^T$, and that $Q(\theta_1, \theta_2) > 0$. Thus the zeroeth, first and second order conditions are met and each set of three points corresponding to each admissible α^* firmly fixes K.

Case 2. $r(0) = 1$, and $|f(\theta)| > 1$ arbitrarily close to 0 for $\theta < 0$. (By to choice of q_1, we also have $|f(\theta)| > 1$ arbitrarily close to 0 for $\theta > 0$.) In this case we shall choose three points to fix K as follows: p_1, p_2 slightly below and above q_1, and p_0 at q_2 (thus disposed counterclockwise). Consider points of concurrence on the ray

$$\{q = \alpha(\cos\pi, \sin\pi)^T = (-\alpha, 0) : \alpha > 0\}.$$

We apply lemma 4.3c(1), 4.3d(1) with $\omega = 0$, $\xi = \pi$ to find there exists $\alpha_0 > 0$ such that for almost every $\alpha^* \in (0, \alpha_0)$

$$i)\ \exists\ \theta_1^* \in (0, \varepsilon) : \phi(\theta_1^*) = \alpha^*, \phi_s(\theta_1^*) > 0,$$
$$ii)\ \exists\ \theta_2^* \in (-\varepsilon, 0) : \phi(\theta_2^*) = \alpha^*, \phi_s(\theta_2^*) < 0.$$

Now let p_1, p_2 be defined by θ_1^*, θ_2^*. Each gives a positive contribution to the second order condition and so does p_0, because the point of concurrence $q = (-\alpha^*, 0)$ is closer to p_0 than the origin. Therefore the three points corresponding to each acceptable α^* firmly fix K.

Case 3. $r(0) = 1$, $|f(\theta)| \equiv 1$ for $\theta \in (-\delta, 0]$. We shall choose points as in Case 1, with $\zeta < 0 \in (-\delta, 0]$ defining p_0, $\xi = \zeta + \pi$. This time, however, we need to estimate the second order condition more carefully. Choose $\varepsilon, \overline{\alpha} > 0$ so that for $\theta_1 \in (0, \varepsilon)$, $\theta_2 \in (\pi - \varepsilon, \pi)$, $\alpha \in (0, \overline{\alpha})$,

$$g^\alpha(\theta_1), g^\alpha(\theta_2) > \frac{1}{2}, \sin(\xi - \theta_1) > \frac{1}{2}\sin\xi, \sin(\theta_2 - \xi) > \frac{1}{2}\sin\xi.$$

Applying the lemma 4.3a(1) and 4.3b(2) with $c > 8\varepsilon(1 + \overline{\alpha})/\sin^2\xi$, respectively to $\omega = 0$ and $\omega = \pi$, we obtain values α^1, α^2 such that, writing $\alpha^0 = \min(\alpha^1, \alpha^2, \overline{\alpha})$, a.a. $\alpha^* \in (0, \alpha^0)$

$$\exists\ \theta_1^* \in (0, \varepsilon) : \phi(\theta_1^*) = \alpha^*, \phi_s(\theta_1^*) > 0$$

and $\exists$ open $S \supset (0, \alpha^0) : \forall\ \alpha \in S$

$$\exists\, \theta_2^* \in (\pi - \varepsilon, \pi) : \phi(\theta_2^*) = \alpha^*, \phi_s(\theta_2^*) < -c\alpha^* .$$

Recall that $a_0 = \sin(\theta_2^* - \theta_1^*)$, $a_1 = \sin(\zeta - \theta_2^*)$, $a_2 = \sin(\theta_1^* - \zeta)$. The second order condition holds a.a. $\alpha \in S$ because by the choice of c

$$\Sigma_0^2\, a_i\, g_i^\alpha \frac{d}{ds}(h_i^\alpha) > \sin(\theta_2^* - \theta_1^*)\,(1 + \alpha^* - (1 + \alpha^*)^2) + \frac{1}{8} c\alpha^* \sin^2\xi$$
$$> \left(\frac{1}{8} c \sin_2 \xi - \varepsilon(1 + \overline{\alpha})\right)\alpha^* > 0.$$

Therefore the three points corresponding to each acceptable α^* firmly fix K. The proof is complete.■

We now prove the two dimensional Kuperberg conjecture.

4.5. Theorem. Any strictly compact 2 dimensional figure with C^2 boundary can be fixed firmly by three points unless it is the disk.

Proof. Let us first assume that ∂K does not admit a thread. We have shown that there are three points fixing ∂K firmly except in the case in which the points previously found to fix K were three points on a maximal inscribed circle. In this case, arbitrarily close to at least one of the three points clockwise and, possibly to another point, counter-clockwise, there must be points at which $|f| > 1$, since otherwise K admits a thread on its boundary. There are several cases which we examine separately.

Case I. There are two different points p_1, p_2 at which $|f|$ is not locally constant on opposite sides, these sides being the same one for each as the side towards which the third point (p_0) lies (along an arc with angle less than π). We call p_0 the reference point in Case I. (In every case the points are named so that they are disposed counterclockwise along the circumference.)

Set the origin at the center of the circle, and let p_0 correspond to $\theta = 0$, p_i to θ_i, $i = 1, 2$. Then $|f| > 1$ for values of θ arbitrarily close to and below θ_1 and arbitrarily close to and above θ_2. We shall slightly shift p_1, p_2 towards p_0, which shall remain fixed. Thus we consider points of concurrence on the ray $\{q = (a, 0) : a > 0\}$. Lemma 4.3 produces shifted points p_1, p_2 satisfying the first order condition each with positive contribution to the second order condition. Since the point of concurrence is closer to p_0 than the origin, at which its contribution is zero, the contribution of p_0 also becomes positive. Thus for almost every q in a neighbourhood of the vertex of the ray triples exist for which the zeroeth, first and second order conditions are met.

Case II. There are two different points p_1, p_2 at which $|f|$ is not locally constant on opposite sides, these sides being the opposite for each as the side towards which the third point (p_0) lies (along an arc with angle less than π). p_0 is the reference point for Case II.

Set the origin at the center of the circle, and let p_0 correspond to $\theta = 0$, p_i to θ_i, $i = 1, 2$. Then $|f| > 1$ for values of θ arbitrarily close to and above θ_1 and arbitrarily close to and below θ_2. We shall slightly shift p_1, p_2 away from p_0, which shall remain fixed. This case is more delicate because the shift tends to make p_0 less gripping, so we must show that there are points p_1 and p_2 whose increased grip counteract this effect. We consider points of concurrence on the ray defined by $\xi = \pi$. An analysis similar to Case 3 of the previous theorem, in which one of the shifted points is chosen with sufficient grip, shows that we may obtain points for which the second order condition is positive.

The remaining cases, with and without threads, shall be reduced to Cases I and II.

Case III. $|f|$ is not locally constant only at one point p_0 (on both sides). Recall p_0, p_1, p_2 are disposed counterclockwise. $|f|$ must remain 1 on the short open arc from p_1 to $-p_0$ (the points p_1' on this arc together with p_0, p_2 fix K), otherwise we would obtain points in Case I (with p_2 the reference point). Similarily $|f| = 1$ on the short open arc joining p_2 to $-p_0$. If the maximal arc containing the short arc joining p_1 to p_2 and having $|f| = 1$ is less than a semi-circumference, we again obtain Case I (with p_0 the reference point). If it is equal to a semi-circumference, let q_1, q_2 be the endpoints. We are now in one of the cases examined in theorem 4.4. If the arc has length greater than π but not 2π, three points can be picked which are in Case II.

No further cases remain, since if at the three points $|f|$ is not locally constant either on one or the other side then we have both cases I and II.

Now let ∂K admit threads, and let p_o, p_1, p_2 lie on arcs of the largest inscribed circle. Suppose the arc in which p_0 lies does not contain p_1, p_2. If the end points closest to p_0 of the arc containing p_1 and the end point closest to p_0 of the arc containing p_2 form and arc opposite p_0 with angle less than π, then we have Case II. If the length equals π, we have one of the cases in theorem 4.4. If the length is greater than π we can pick points in Case III. There remains the case in which the three points lie on the same arc, which must have length greater than π. It is easy to generate Case II, unless K is the unit disk.■

Appendix

A. Calculating Closed Strictly Compact Curves

Let $K \subset \mathbb{R}^2$ be a connected compact figure with C^2 boundary of length L. Let ∂K be the closed C^2 curve which is the boundary of the figure. Let $N : \partial K \to S^1$ be the Gauss Map. We assume that the curvature is strictly positive. Let us write

$$f: S^1 \to \partial K \subseteq \mathbb{R}^2 \tag{A.1}$$

for the C^2 inverse of N. We shall use the angle θ to denote points in S^1, and refer to its as the Gauss coordinate. Then

$$N = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad T = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{A.2}$$

As usual, defining an arc length parameter s which may satisfy $s = 0$ at $\theta = 0$ and $\frac{\mathrm{d}s}{\mathrm{d}\theta} > 0$, we may write

$$\frac{\mathrm{d}N}{\mathrm{d}s} = kT, \quad \frac{\mathrm{d}T}{\mathrm{d}s} = -kN. \tag{A.3}$$

where k is the curvature. Combining A.2 and A.3,

$$\frac{\mathrm{d}N}{\mathrm{d}s} = T\frac{\mathrm{d}\theta}{\mathrm{d}s} \Leftrightarrow k = \frac{\mathrm{d}\theta}{\mathrm{d}s}. \tag{(A.4)}$$

Define now the functions g and h (of s or θ) by

$$f = gN + hT. \tag{A.5}$$

Then

$$T = \frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(gN + hT) = \left(\frac{\mathrm{d}g}{\mathrm{d}s} - kh\right)N + \left(\frac{\mathrm{d}h}{\mathrm{d}s} + kg\right)T. \tag{A.6}$$

Hence we have a system of o.d.e.'s for g and h,

$$\frac{\mathrm{d}g}{\mathrm{d}s} = kh, \quad \frac{\mathrm{d}h}{\mathrm{d}g} = 1 - kg, \tag{A.7}$$

also

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|f|^2 = gg' + hh' = kgh + h(1 - kg) = h. \tag{A.8}$$

We can change the variables in equations A.7 to θ. Defining

$$r = k^{-1} = \frac{\mathrm{d}s}{\mathrm{d}\theta} \tag{A.9}$$

to be the radius of curvature, we obtain the system

$$\frac{\mathrm{d}g}{\mathrm{d}\theta} = h, \frac{\mathrm{d}h}{\mathrm{d}\theta} = r - g\,, \tag{A.10}$$

Since the curve is closed we have some integral identities:

$$\int_0^L h\mathrm{d}s = \int_0^L \frac{1}{2}\ (|f|^2)'\mathrm{d}s = \left[\frac{1}{2}|f|^2\right]_0^L = 0 \tag{A.11}$$

$$\int_0^L k\mathrm{d}s = \int_0^L \frac{\mathrm{d}\theta}{\mathrm{d}s}\,\mathrm{d}s = [\theta]_0^L = 2\pi \tag{A.12}$$

$$\int_0^{2\pi} h\mathrm{d}\theta = \int_0^{2\pi} \frac{\mathrm{d}g}{\mathrm{d}\theta}\,\mathrm{d}\theta = [g]_0^{2\pi} = 0 \tag{A.13}$$

$$\int_0^{2\pi} g\mathrm{d}\theta = \int_0^L kg\mathrm{d}s = \int_0^L \left(1 - \frac{\mathrm{d}h}{\mathrm{d}s}\right)\mathrm{d}s = L - [h]_0^L = L \tag{A.14}$$

$$\int_0^{2\pi} r\mathrm{d}\theta = \int_0^{2\pi} \frac{\mathrm{d}s}{\mathrm{d}\theta}\,\mathrm{d}\theta = L. \tag{A.15}$$

We may integrate equations A.10, which may be written

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{pmatrix} g \\ h \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}. \tag{A.16}$$

Let us set $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, for which

$$\frac{\mathrm{d}P}{\mathrm{d}\theta} P^{-1} = P^{-1}\frac{\mathrm{d}P}{\mathrm{d}\theta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.17}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(P\begin{pmatrix} g \\ h \end{pmatrix}\right) = P\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{pmatrix} g \\ h \end{pmatrix} + \frac{\mathrm{d}P}{\mathrm{d}\theta}\begin{pmatrix} g \\ h \end{pmatrix} = P\begin{pmatrix} 0 \\ r \end{pmatrix}, \tag{A.18}$$

so that

$$\left[P\begin{pmatrix}g\\h\end{pmatrix}\right]_0^\theta = \int_0^\theta P(\omega)\begin{pmatrix}0\\r(\omega)\end{pmatrix}d\omega = \int_0^\theta r(\omega)\begin{pmatrix}-\sin\omega\\\cos\omega\end{pmatrix}d\omega. \tag{A.19}$$

g and h are periodic (corresponding to a closed curve) if and only if

$$\int_0^{2\pi} r(\omega)\begin{pmatrix}-\sin\omega\\\cos\omega\end{pmatrix}d\omega = 0. \tag{A.20}$$

The homogeneous solutions of equation A.15 are

$$\begin{pmatrix}g_0\\h_0\end{pmatrix} = a\begin{pmatrix}-\cos(\theta-\theta_0)\\\sin(\theta-\theta_0)\end{pmatrix}. \tag{A.21}$$

These correspond to a change of origin to the point $a(cos\theta_0, \sin\theta_0)^T$ since adding them to the original solution displaces f by

$$g_0 N + h_0 T = a\left\{\sin(\theta-\theta_o)\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix} - \cos(\theta-\theta_0)\begin{pmatrix}-\cos\theta\\\sin\theta\end{pmatrix}\right\} = -a\begin{pmatrix}\cos\theta_0\\\sin\theta_0\end{pmatrix}.$$

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