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CYCLIC PRICING BY A DURABLE GOODS MONOPOLIST: CORRIGENDUM

Abstract

In this paper we present and correct some important errors found in Conlisk , Gerstner and Sobel (1984). whose corrections drastically modify the principal message of that paper. They propose a model in discrete time. such that at each time a new cohort of agents enters the market (each cohort is composed by two types of agents. high value and low value agents), and a monopolist offering a durable good. They argue that in this model the monopolist will charge a cyclic price path as a subgame perfect equilibrium. Instead of that. we show that either the monopolist charge a single price forever as a subgame perfect equilibrium or. a subgame perfect equilibrium does not exist.

Resumen

En este trabajo presentamos y corregimos algunos errores importantes encontrados en el estudio de Conlisk , Gerstner and Sobel (1984). cuyas correcciones modifican fundamentalmente los resultados. Ellos proponen un modelo en tiempo discreto. tal que en cada período entra una nueva generación de agentes (cada generación está compuesta de dos tipos de consumidores. los de valoración alta y los de baja), y un monopolista ofreciendo un bien durable. Ellos argumentan que en ese modelo el monopolista cargará una senda de precios cíclica como un equilibrio perfecto en subjuegos. En vez de eso. nosotros probamos que siempre cargará un precio fijo como equilibrio perfecto en subjuegos. o simplemente no existe ningún equilibrio perfecto en subjuegos.

I) Introduction

In the paper by Conlisk, Gerstner and Sobel (1984) there are important errors whose corrections drastically modify the principal results given in that paper, leading to opposite conclusions. These errors and the corresponding corrections, summarized in our Theorem 1, are shown in the sequel. For the details and formulation of the model, see Conlisk *et. al.*

I) It is argued, in Section III of that paper, that the monopolist's pricing strategy must in equilibrium (a subgame perfect equilibrium: In Conlisk *et. al*, page 496, it is affirmed:'The n^*-equilibrium is always subgame perfect.') involve a cyclic price path characterized by

$$p_j = (1 - \beta^{n-j})V_1 + \beta^{n-j}V_2 \tag{1}$$

for $j = 1, ..., n$ (for some n). This statement is false, in virtue of the following propositions.

a) If $\alpha V_1 > V_2$, then a subgame perfect equilibrium does not exist, even accepting the no-commitment assumption. Therefore, the unique theorem in Conlisk *et. al* is false, because for no $n \geqq 1$, can the prices given by (1) charged forever a subgame perfect equilibrium be.

b) If $\alpha V_1 \leq V_2$, then there is a unique subgame perfect equilibrium, that in which the monopolist charges V_2 forever and all consumers, high value and low value, buy the good at the moment they enter the market. Therefore, no cyclic behavior is obtained in this model.

c) According to Conlisk *et. al*, the number n^*, presumably characterized in Section IV as a subgame perfect equilibrium, need not be the largest element of the one-man, the largest element of the first column of the matrix II, due to the no-commitment assumption. This statement is false: If it were the case that n^* does not coincide with the largest element of the one-man, then the prices given by (1) with n^* would not even be a Nash equilibrium strategy, and therefore those strategies would not be a subgame perfect equilibrium. In other words, if prices given by (1) with some n are subgame perfect equilibrium, they must be a Nash equilibrium, so n must be the largest element of the one-man. Furthermore, if n^* would not coincide with the largest element of the one-man, then at any time of the form n^*k+1 with $k \in \{0, 1, 2, ..\}$, the (n^*k+1)-man would not be maximizing benefits, because at these times, the corresponding subgames are exactly the same as the game at the onset. If the one-man is not maximizing, then for any $k \in \{0, 1, 2, ..\}$, the $(n^*k + 1)$-man is not maximizing. Obviously, this argument does apply even accepting the no-commitment assumption in both cases, when $\alpha V_1 > V_2$ and when $\alpha V_1 \leq V_2$.

In what follows we present the arguments proving our statements (a) and (b) in I. At the end of Section II we state Theorem 1. Finally, in Section III, we present the conclusions and some comments.

II) Proof of the claims in (I)

a) If $\alpha V_1 > V_2$, then a subgame perfect equilibrium does not exist, with or without the no-commitment assumption.

Proof: First of all, we express what we understand by the no-commitment assumption by means of the following

Definition 1 *The no-commitment assumption is the imposition that the monopolist never can commit himself to a given strategy once-and-for-all, unless this strategy is sustainable if in future times the monopolist is able to revise it. Formally, only subgame perfect equilibrium strategies are considered 'equilibrium strategies.'*

Now, we denote by $\pi(V_1)$ the present value of the monopolist's stream from time 1 to infinity if V_1 is charged forever, we have

$$\pi(V_1) = \frac{N\alpha V_1}{1-\rho}.$$

Similarly, we denote by $\pi(n, 1, \beta, \rho)$ (in the notation of Conlisk *et. al* we have $\pi(n, 1, \beta, \rho) = \pi(n, 1)$) the present value of the monopolist's total stream from time 1 to infinity if prices $\{p_j\}$ as given in (1) are charged forever for some n. We have

$$\pi(n,1,\beta,\rho) = \frac{N}{1-\rho^n}\left\{\alpha\left[\sum_{j=1}^{n}((1-\beta^{n-j})V_1 + \beta^{n-j}V_2)\rho^{j-1}\right] + nV_2(1-\alpha)\rho^{n-1}\right\}$$

The term $\alpha N\left[\sum_{j=1}^{n}((1-\beta^{n-j})V_1 + \beta^{n-j}V_2)\rho^{j-1}\right] + nNV_2(1-\alpha)\rho^{n-1}$ is denoted by $R(n, 1, \beta, \rho)$ ($R(n, 1, \beta, \rho) = R(n, 1)$ in the notation of Conlisk *et. al*), which is the present value of the monopolist's profit stream as calculated from the first period to the nth period of the cycle. Therefore $\pi(n, 1, \beta, \rho) = \frac{1}{1-\rho^n}R(n, 1, \beta, \rho)$.

Rearranging,

$$\pi(n,1,\beta,\rho) = \frac{N\alpha V_1}{1-\rho} + \frac{N}{1-\rho^n}\left\{nV_2(1-\alpha)\rho^{n-1} - \alpha(V_1 - V_2)\left[\frac{1-(\frac{\rho}{\beta})^n}{1-(\frac{\rho}{\beta})}\right]\beta^{n-1}\right\} \tag{2}$$

We demonstrate this equality in the Appendix.

Now, if $\beta = \rho$, we have

$$\pi(n,1,\rho,\rho) = \frac{N\alpha V_1}{1-\rho} + \frac{N}{1-\rho^n}\left\{nV_2(1-\alpha)\rho^{n-1} - n\alpha(V_1 - V_2)\rho^{n-1}\right\},$$

which results in

$$\pi(n,1,\rho,\rho) = \frac{N\alpha V_1}{1-\rho} + \frac{nN\rho^{n-1}}{1-\rho^n}\left\{V_2 - \alpha V_1\right\}. \tag{3}$$

The expression (3) is the key element for the analysis of the model.

Clearly, the monopolist would only choose cyclic prices at time one if $\pi(n, 1, \rho, \rho) \geq \pi(V_1)$, that is, only if $V_2 \geq \alpha V_1$; therefore, if $V_2 < \alpha V_1$ he would never choose to charge prices as given in (1) at time one.

Thus, if we do not consider the no-commitment assumption, we have that the strategy charging V_1 forever is the unique best strategy at this time. Indeed, at time one the monopolist has two possibilities: To charge V_1 forever or not; now, if he does not decide to charge V_1 forever, in principle, he would consider the benefits given by $\pi(n, 1, \rho, \rho)$ for some n (the largest of those, if it exists), due to that the consumer's surplus is exploited at the maximum possible (high value consumers would never buy today at a price equal to V_1 if they expect a sale sooner or later), but since $\pi(n, 1, \rho, \rho) < \pi(V_1)$ for all n, the best the monopolist can do is to charge at time one V_1 forever. Therefore, to charge V_1 forever is the unique Nash equilibrium. On the other hand, as time goes on the accumulation of low value consumers will make it profitable for the monopolist to charge V_2 sooner of later. This implies that when $V_2 < \alpha V_1$ a subgame perfect equilibrium does not exist, if we do not consider the no-commitment assumption.

Now, let's assume that to charge V_1 forever is ruled out by assumption. That is, we take into account the no-commitment assumption.

In order to prove our statement, it suffices to note that for any $n \geqq 1$, we have $\pi(n+1, 1, \rho, \rho) > \pi(n, 1, \rho, \rho)$. This follows directly from the inequality (13) in Conlisk *et. al.* We can explicitly prove this using our function f defined in (b) and noting that f' in (4) is always negative.

Indeed, given this result, if we take the number n^* given in the unique theorem in Conlisk *et. al,* the strategy prices given by (1) with n^* as the period length cannot be a subgame perfect equilibrium, because not only the one-man is not maximizing $(\pi(n^* + 1, 1, \rho, \rho) > \pi(n^*, 1, \rho, \rho))$, but also at any time of the form $n^*k + 1$ with $k \in \{0, 1, 2, ..\}$, the $(n^*k + 1)$-man is not maximizing benefits, because at these times, the corresponding subgames are exactly the same as the game at the onset. If the one-man is not maximizing, then for any $k \in \{0, 1, 2, ..\}$, the $(n^*k + 1)$-man is not maximizing.

Therefore, we have proven that if $V_2 < \alpha V_1$, then for no n can the prices given by (1) a subgame perfect equilibrium be, so the number n^* given in Conlisk *et. al*, is not a subgame perfect equilibrium.

It is very important to notice, however, that this reasoning does not depend on the form of the no-commitment assumption. It is the consequence way that the monopolist and the consumers evaluate their decisions (their pay-off functions), a process that cannot be modified by the no-commitment assumption, given the approach used in Conlisk *et al.* That is, to obtain a subgame perfect equilibrium in this case, we must modify the behavior of the consumers and the monopolist.

This concludes the proof of (a). ■

b) If $V_2 \geqq \alpha V_1$, then the price strategy $p_t = V_2$ for all $t \geqq 1$ and all consumers buying the good at the moment they enter the market, is the only one subgame perfect equilibrium in this model. In particular, this implies that for no $n > 1$, can the prices given by (1) a subgame perfect equilibrium be.

Proof: Suppose then that $V_2 > \alpha V_1$ and $\rho < 1$. We recall that the present value benefits at time one if prices as in (1) for some n are charged forever, are given by:

$$\pi(n, 1, \rho, \rho) = \frac{N\alpha V_1}{1-\rho} + \frac{nN\rho^{n-1}}{1-\rho^n}\{V_2 - \alpha V_1\}.$$

Notice that $\pi(1, 1, \rho, \rho) = \frac{NV_2}{1-\rho}$, and therefore, to charge V_2 forever is exactly the same as charging prices given by (1) with n equals 1 forever. Now, we consider the function $f(x) = \frac{x\rho^x}{1-\rho^x}$. Then,

$$f'(x) = \frac{\rho^x}{(1-\rho^x)^2}\left[1 - \rho^x + \ln \rho^x\right], \tag{4}$$

and note that $f'(x) < 0$, if and only if $1 + \ln \rho^x < \rho^x$. Now, it is straightforward to prove that $1 + \ln \rho^x < \rho^x$ for all $x > 0$ and $1 + \ln \rho^x = \rho^x$ if and only if $x = 0$ or $\rho = 1$.

Therefore, we have that $f'(x) < 0$ for all $x > 0$ and hence, we have that

$$\pi(1,1,\rho,\rho) \geqq \pi(n,1,\rho,\rho)$$

for all $n \geq 1$ and

$$\pi(1,1,\rho,\rho) > \pi(n,1,\rho,\rho) \tag{5}$$

if $n > 1$.

Take any time in the game, and the corresponding subgame. Notice that any subgame, with the strategy charging V_2 forever, is identical to the game at time one because there are no low consumers accumulated. Therefore, if we show that to charge V_2 forever is the unique best strategy at time one, it will be the best (the only one) at any subgame, and our claim will be proven.

Now, at time one, the monopolist can decide to charge V_1 forever or not. If he charges V_1 forever he gets $\frac{N\alpha V_1}{1-\rho}$, and if he charges V_2 forever he gets $\frac{NV_2}{1-\rho} > \frac{N\alpha V_1}{1-\rho}$. Therefore to charge V_2 forever dominates the other strategy. On the other hand, if the monopolist does not decide to charge V_1 forever, *a priori,* the best he can do is to charge a cyclic price strategy given by (1) with the appropriate n: Indeed, if he does not charge V_1 forever, he would plan to make a sale sooner or later, and in this case, he would charge a cyclic strategy forever with n (because it exploits the consumers' surplus at the maximum possible) such that generates the largest $\pi(n,1,\rho,\rho)$ among all n, that is $n = 1$, due to that $\pi(1,1,\rho,\rho) > \pi(n,1,\rho,\rho)$ for all n. Thus, to charge V_2 forever is the best strategy at time one (if one would prefer a more explicit argument, it is also easy to show, by means of a direct comparison, that to charge V_2 forever dominates not only those strategies charging the same cyclic path forever, but also those in which the monopolist consider to charge different cycles one after the other).

The uniqueness of this equilibrium follows directly from the strict inequality (5).

Now we consider the case when $V_2 - \alpha V_1 = 0$. First, the fact that to charge V_2 forever is a subgame perfect equilibrium. The proof of this is analogous to the one above and hence is omitted.

Now we will prove that given any $n > 1$, then prices $\{p_l(n)\}_{l=1}^n$ are not subgame perfect equilibrium.

To this end, we will show that for any $k \in \{0,1,2...\}$ and any t of the form $t = kn + j$ with j satisfying $2 \leq j \leq n$, there exists a strategy that from j henceforth dominates the original one.

Therefore, take one t as defined above, that is, t is any period that is not a starting period of a cycle. Let's consider the benefits that the monopolist receives if he does not change the strategy decided at time one from time t henceforth, that is, if he charges $p_j(n)$ at time $kn+j$, $p_{j+1}(n)$ at time $kn+j+1$, and so on. We denote by $\pi_{kn+j}(n,1,\rho,\rho)$

the present value of these benefits. Then

$$\pi_{kn+j}(n,1,\rho,\rho) = \rho^{1-j}\left[R(n,1,\rho,\rho) - \alpha N\sum_{l=1}^{j-1} p_l(n)\rho^{l-1}\right] + \rho^{n-j+1}\frac{R(n,1,\rho,\rho)}{(1-\rho^n)}. \tag{6}$$

We prove this in the Appendix.

Rearranging this last equality, we have

$$\pi_{kn+j}(n,1,\rho,\rho) = \rho^{1-j}\left[\frac{R(n,1,\rho,\rho)}{(1-\rho^n)} - \alpha N\sum_{l=1}^{j-1} p_l(n)\rho^{l-1}\right] \tag{7}$$

Now consider an alternative strategy as follows: To start again from t a new period cycle $\{p_l(\hat{n})\}_{l=1}^{\hat{n}}$ for some $\hat{n}$.

Then, in order to prove our affirmation, we compute the present value of the benefits for the monopolist if from t he decides to charge a new cycle $\{p_l(\hat{n})\}_{l=1}^{\hat{n}}$ for some $\hat{n}$. Denoting by π_{aj} its benefits, we have

$$\pi_{aj} = \frac{R(\hat{n},1,\rho,\rho)}{(1-\rho^{\hat{n}})} + (j-1)N(1-\alpha)V_2\rho^{\hat{n}-1}. \tag{8}$$

We demonstrate this in the Appendix.

Recall that $\frac{R(\hat{n},1,\rho,\rho)}{(1-\rho^{\hat{n}})} = \frac{N\alpha V_1}{1-\rho}$ for any $\hat{n}$, so we take $\hat{n} = 1$ in order to obtain the best alternative at this time. Therefore, the present value of the benefits with $\hat{n} = 1$ is

$$\pi_{aj} = \frac{N\alpha V_1}{1-\rho} + (j-1)N(1-\alpha)V_2$$

Now we will show that $\pi_{aj} > \pi_{kn+j}(n,1,\rho,\rho)$ and therefore the proof of claim (b) will be completed.

We have $\pi_{aj} > \pi_{kn+j}(n,1,\rho,\rho)$ if and only if

$$\frac{N\alpha V_1}{1-\rho} + (j-1)N(1-\alpha)V_2 > \rho^{1-j}\left[\frac{R(n,1,\rho,\rho)}{(1-\rho^n)} - N\alpha\sum_{l=1}^{j-1} p_l(n)\rho^{l-1}\right]. \tag{9}$$

Recall again that $\frac{R(n,1,\rho,\rho)}{(1-\rho^n)} = \frac{N\alpha V_1}{1-\rho}$. Hence, replacing, the inequality (9) is equivalent to

$$\frac{N\alpha V_1}{1-\rho} + (j-1)N(1-\alpha)V_2 > \rho^{1-j}\left[\frac{N\alpha V_1}{1-\rho} - N\alpha\sum_{l=1}^{j-1} p_l(n)\rho^{l-1}\right],$$

and thus, equivalent to

$$N\alpha\sum_{l=1}^{j-1} p_l(n)\rho^{l-1} + \rho^{j-1}(j-1)N(1-\alpha)V_2 > N\alpha V_1\frac{(1-\rho^{j-1})}{1-\rho}. \tag{10}$$

Now, taking the left side of this inequality $N\alpha\sum_{l=1}^{j-1} p_l(n)\rho^{l-1} + \rho^{j-1}(j-1)N(1-$

$\alpha)V_2$, we have that

$$N\alpha\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}+\rho^{j-1}(j-1)N(1-\alpha)V_2 > N\alpha\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}+\rho^{j-2}(j-1)N(1-\alpha)V_2 \tag{11}$$

because $0<\rho<1$. Now, observing that if we denote by $\{p_l(j-1)\}_{l=1}^{j-1}$ the prices given by (1) with $j-1$ as the period length, we have that

$$p_l(n) > p_l(j-1) \text{ for all } l=1,..,j-1, \tag{12}$$

because $j-1<n$. We demonstrate this in the Appendix.

Therefore, taking the right side of (11), we have

$$N\alpha\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}+(j-1)N(1-\alpha)V_2\rho^{j-2} > N\alpha\sum_{l=1}^{j-1}p_l(j-1)\rho^{l-1}+(j-1)N(1-\alpha)V_2\rho^{j-2},$$

and the right side of this last equality is exactly

$$R(j-1,\rho,\rho)=N\alpha V_1\frac{(1-\rho^{j-1})}{(1-\rho)},$$

that is, we have shown that

$$N\alpha\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}+(j-1)N(1-\alpha)V_2\rho^{j-2} > N\alpha V_1\tfrac{(1-\rho^{j-1})}{1-\rho}. \tag{13}$$

Now, recalling the inequality (11), we have

$$N\alpha\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}+(j-1)N(1-\alpha)V_2\rho^{j-1} > N\alpha\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}+(j-1)N(1-\alpha)V_2\rho^{j-2} > N\alpha V_1\tfrac{(1-\rho^{j-1})}{1-\rho},$$

and therefore (the last inequality is due to (13)),

$$N\alpha\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}+(j-1)N(1-\alpha)V_2\rho^{j-1} > N\alpha V_1\tfrac{(1-\rho^{j-1})}{1-\rho},$$

which is exactly the inequality (10), and hence we have proven that $\pi_{aj}>\pi_{kn+j}(n,1,\rho,\rho)$.

This concludes the proof of claim (b).■

Notice that our efforts to present formal arguments for our propositions have led us to prove strong statements that we summarize in the following

Theorem 1 *If $\beta=\rho$, then*

a) If $\alpha V_1>V_2$ and we do not consider the no-commitment assumption, then there is a unique Nash equilibrium whose strategies are: The monopolist charges V_1 forever,

high consumers buy at the moment they enter the market and low consumers do not buy at all. This Nash equilibrium is not a subgame perfect equilibrium.
On the other hand, even if we assume the no-commitment hypothesis, then for no $n \geqq 1$, can the prices given by (1) a subgame perfect equilibrium be, and no Nash equilibrium exist.
b) If $\alpha V_1 \leq V_2$, then there is a unique Nash equilibrium which is also a subgame perfect equilibrium, whose strategies are: The monopolist charges V_2 forever, and high and low consumers buy at the moment they enter the market.

III) Conclusions

Theorem 1 is a complete characterization of the possible equilibrium strategies in the model when $\beta = \rho$.

Surprisingly enough, and in sharp contrast to the conclusions in Conlisk *et. al*, the conclusion is that in this model, when $\beta = \rho$, there is no cyclic optimal pricing strategy by the monopolist. We stress here, once again, that this result does not depend on the precise way that Conlisk *et. al* state the no-commitment assumption. To modify the result, we must modify the pay-off functions.

At first glance, our conclusions may appear paradoxical. *A priori*, it is strange not to obtain cyclic behavior from the monopolist. In relation to this paradoxical fact we have to divide the analysis into two principal cases: When $\alpha V_1 \leq V_2$ and when $\alpha V_1 > V_2$.

First, let's examine the case when $\alpha V_1 \leq V_2$. Here we do not think that the result is necessarily paradoxical. Although it depends heavily on the way that the model describes the behavior of the consumers, that is, it depends heavily on the exact form of the prices, we conjecture that if we would model the consumers' behavior in another way, we would obtain the same result. This intuition is due to the fact that it is never profitable for the monopolist to charge a higher price than V_2. That is, there is no trade off between charging higher prices than V_2 and accumulating low value agents some periods to receive the gains of their purchases later, and to charge V_2 every period: It is always better to charge V_2 forever. This intuition is also backed by the fact that this equilibrium is the unique subgame perfect equilibrium in this model.

Second, is the case when $\alpha V_1 > V_2$. In this case, the result is not the one that people would anticipate in real life or, more precisely, cyclic behavior should be the result of a good model. Indeed, in a representative model, we would expect the monopolist to charge V_1 at some times cyclically and optimally . The intuition here is quite clear: The best the monopolist can do at the outset is to charge V_1 forever, but the accumulation of low value agents makes it lucrative for him to drop the price sooner or later. A good model should support this intuition as a subgame perfect equilibrium.

This result then, more than being paradoxical, reflects a weakness of the model.

Appendix

I) Proof of (2):

By definition of prices in (1), we have

$$\pi(n,1,\beta,\rho) = \frac{N}{1-\rho^n}\left\{\alpha\left[\sum_{j=1}^{n}((1-\beta^{n-j})V_1+\beta^{n-j}V_2)\rho^{j-1}\right]+ nV_2(1-\alpha)\rho^{n-1}\right\},$$

so

$$\pi(n,1,\beta,\rho) = \frac{N}{1-\rho^n}\left\{\alpha\sum_{j=1}^{n}V_1\rho^{j-1}-\alpha\sum_{j=1}^{n}\rho^{j-1}\beta^{n-j}V_1+ \alpha\sum_{j=1}^{n}\beta^{n-j}V_2\rho^{j-1}+nV_2(1-\alpha)\rho^{n-1}\right\},$$

then

$$\pi(n,1,\beta,\rho) = \frac{N}{1-\rho^n}\left\{\alpha V_1\frac{1-\rho^n}{1-\rho}+nV_2(1-\alpha)\rho^{n-1}- \alpha\left\{V_1-V_2\right\}\beta^{n-1}\sum_{j=0}^{n-1}\left(\frac{\rho}{\beta}\right)^j\right\},$$

hence

$$\pi(n,1,\beta,\rho) = \frac{N\alpha V_1}{1-\rho}+\frac{NnV_2}{1-\rho^n}(1-\alpha)\rho^{n-1}- \frac{N\alpha}{1-\rho^n}\left\{V_1-V_2\right\}\left[\frac{1-(\frac{\rho}{\beta})^n}{1-(\frac{\rho}{\beta})}\right]\beta^{n-1},$$

which is precisely the equation (2)

2) Proof of (6):

Take any $k\in\{0,1,2...\}$ and any t of the form $t=nk+j$ with j satisfying $2\leq j\leq n$. Now let $\{p_l(n)\}_{l=1}^{n}$ the prices given by (1). Therefore

$$\begin{gathered}\pi_{kn+j}(n,1,\rho,\rho) = \alpha N\left[p_j(n)+p_{j+1}(n)\rho+p_{j+2}(n)\rho^2+.+V_2\rho^{n-j}\right]+ \\ (1-\alpha)NnV_2\rho^{n-j}+\left[\alpha N(p_1(n)\rho^{n-j-1}+..+V_2\rho^{2n-j})+(1-\alpha)nNV_2\rho^{2n-j}\right] \\ +\sum_{l=l}^{\infty}\rho^{nl}\left[\alpha N(p_1(n)\rho^{n-j+1}+..+V_2\rho^{2n-j})+(1-\alpha)nNV_2\rho^{2n-j}\right]\end{gathered},$$

so

$$\begin{gathered}\pi_{kn+j}(n,1,\rho,\rho) = \alpha N\left[p_j(n)+p_{j+1}(n)\rho+p_{j+2}(n)\rho^2+..+V_2\rho^{n-j}\right] \\ +(1-\alpha)NnV_2\rho^{n-j}+\frac{\rho^{n-j+1}}{1-\rho^n}R(n,1,\rho,\rho)\end{gathered}. \tag{14}$$

Now, observe that

$$\begin{gathered}\alpha N\left[p_j(n)+p_{j+1}(n)\rho+p_{j+2}(n)\rho^2+..+V_2\rho^{n-j}\right]+(1-\alpha)NV_2\rho^{n-j} = \\ \rho^{1-j}\left[R(n,1,\rho,\rho)-\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}\right]\end{gathered},$$

therefore the equation (14) becomes

$$\pi_{kn+j}(n,1,\rho,\rho) = \rho^{1-j}\left[R(n,1,\rho,\rho)-\sum_{l=1}^{j-1}p_l(n)\rho^{l-1}\right]+\frac{\rho^{n-j+1}}{1-\rho^n}R(n,1,\rho,\rho),$$

which is precisely the equation (5).

3) Proof of (8):

We have to prove that $\pi_{aj}=\frac{R(\hat{n},1,\rho,\rho)}{(1-\rho^{\hat{n}})}+(j-1)N(1-\alpha)V_2\rho^{\hat{n}-1}$.

Notice that until the period j, there are $j-1$ generations of low value consumers accumulated. Now if the monopolist starts a new cycle with period $\hat{n}$, then $\hat{n}-1$ periods later he will earn the present value of those $j-1$ generations accumulated before he started the new period, that is $(j-1)N(1-\alpha)V_2\rho^{\hat{n}-1}$, plus the normal present value of the new period cycle, that is, $\frac{R(\hat{n},1,\rho,\rho)}{(1-\rho^{\hat{n}})}$, therefore the present value from j is $\frac{R(\hat{n},1,\rho,\rho)}{(1-\rho^{\hat{n}})} + (j-1)N(1-\alpha)V_2\rho^{\hat{n}-1}$, and thus we have proven the statement.

4) Proof of (12):

By definition, $p_l(n) = (1-\beta^{n-l})V_1 + \beta^{n-l}V_2$ for all $l \leq n$. Now, consider the function $h(x) = (1-\beta^{x-l})V_1 + \beta^{x-l}V_2$ for $x \geqq 0$. We have $h'(x) = (V_2 - V_1)\beta^{x-l}\ln\beta$. Therefore, we have that $h'(x) > 0$ for all $x \geqq 0$. This concludes the proof of (11).

References

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