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# CHAOS VS. PATIENCE IN MACROECONOMIC MODELS OF CAPITAL ACCUMULATION: NEW APPLICATIONS OF A UNIFORM NEIGBOHOOD TURNPIKE THEOREM

***Abstract***

We present in this paper some new results in relation to the strong incompatibility between chaos and patience in Macroeconomic models of capital accumulation. These results are both explicit and non-trivial applications of the general theorem proven in Guerrero-Luchtenberg (2000), in which the statement (Theorem 2) 'chaos vanishes as the discount factor tends to one, ' is formally presented. Here, we show precisely how this statement applies to some important indicators of chaos, not analyzed before. Furthermore, we will show that, for a given family of optimal growth problems, there is a bound on the discount factor $\delta'$ such that, for any $\delta$ larger than $\delta'$, any type of chaos is negligible.



***Resumen***

Presentamos en este trabajo algunos nuevos resultados respecto de la fuerte incompatibilidad entre fenómenos caóticos y el grado de paciencia en modelos macroeconómicos de acumulación de capital. Estos resultados son explícitas y no triviales aplicaciones del teorema general probado en Guerrero-Luchtenberg (2000), en el cual un enunciado del tipo 'el caos tiende a desaparecer cuando el factor de descuento tiende a cero,' es formalmente presentado. Aquí, mostramos en forma precisa cómo dicho teorema es aplicado a algunos importantes indicadores de caos no analizados con anterioridad. Más aun, mostraremos que, para una dada familia de problemas de crecimiento óptimo, existe un cota sobre el factor de descuento $\delta'$ de modo que, para cualquier $\delta$ mayor $\delta'$, cualquier tipo de caos es irrelevante.

## *Introduction*

Consider a deterministic, reduced form of optimal growth models with discounting, as follows:

$$\begin{aligned}&\sup \sum_{t=0}^{\infty} \delta^{t+1} u(x_t, x_{t+1})\\ &\text{s. t. } (x_t, x_{t+1}) \in D \text{ for all } t \geq 0\\ &x_0 \text{ given,}\end{aligned}$$

where $D$ is the feasible set, $x_0$ is the initial state, $\delta$ is the discount factor (a real number between zero and one) and $u$ is the felicity function. [1]

It is well known that, in this type of models, chaos is precluded under strong concavity assumptions over the felicity function, if the felicity function is fixed (the standard turnpike theorems; see, specially, McKenzie (1986) and Sheinkman 1976). More precisely, if the discount factor is large enough, the optical path of capital accumulation converges to the steady state, *given the felicity function.* On the other hand, it is also well known that chaos is possible in that type of models. See, for example, Boldrin and Montrucchio (1986) and, specially, Nishimura and Yano (1995) and Nishimura, Sorger and Yano (1994), provided that they show families of strictly concave felicity functions, such that for any value of the discount factor, there is a member of the family which displays chaos, highlighting the necessity of an appropriate justification of the uniform comparative analysis used in empirical works.

Indeed, in applications, the set-up is typically a family of models instead of a single model, in such a way that felicity functions are not fixed. Therefore, the standard turnpike theorems cannot be cited in order to ensure the existence of a single value of the discount factor such that there is convergence to the steady states all over the family. Hence, the typical dynamic comparative analysis, by means of the steady states, cannot be used without further justification.

To justified that type of comparative analysis, we may cite some works that show the existence of upper bounds for the discount factor in order to a given type of chaos to be possible, for instance, Mitra (1996) and (1998), Montrucchio and Sorger (1996), Nishimura and Yano (1995), and Sorger (1994), among others. Nevertheless, in all these studies before named, the type of chaos, as commented, is fixed, and hence the upper bounds cannot preclude other types of chaos.

On the other hand, in order to find a global justification of the comparative analysis we can appeal to the uniform turnpike theorem proven in Guerrero-Luchtenberg (2000), the theorem 3 in that paper. That theorem, nonetheless, is proven under strong assumptions of the type 'uniform strong concavity over the family,' (assumption A'7 in that paper), which can be notably relaxed and still obtain a result that can be interpreted as 'quasi uniform convergence to the steady states all over the family': The uniform neighborhood turnpike theorem in Guerrero-Luchtenberg (2000). This last result, however, has the counterpart fact that it does not explicitly show how the chaos is precluded.

[1] Note that the model presented here is much more general than the Ramsey-Solow model of capital accumulation.

For this reason, in Guerrero-Luchtenberg (2000), the case of the ergodic chaos is treated in detail. Furthermore, in that paper is suggested that the uniform neighborhood turnpike theorem can be also used to explicitly rule out other type of chaos.

The purpose of this study is, therefore, to show explicitly how the theorem 2 in Guerrero-Luchtenberg (2000) is applied in order to preclude some well known and important indicators of chaos. Furthermore, we will show that, for a given family of optimal growth model, there is a bound on the discount factor $\hat{\delta}$ such that, for any $\delta > \hat{\delta}$, any type of chaos is negligible all over the family, providing then of a general justification for the comparative analysis used in empirical works.

The rest of the paper is as follows. In section 2, for the sake of completeness, we roughly introduce the model and the basic definitions, and we announce the theorem 2 given inGuerrero-Luchtenberg (2000) . Section 3 present some basic definitions about dynamical systems, and we prove our theorem 4, which will notably simplifies the proofs of the results that are the main objective of this paper, the theorems 5, 6 and 7. In Section 4 we present the proofs our main results. Finally, we conclude in section 5.

## *The model*

As our work is heavily based on Guerrero-Luchtenberg (2000) and McKenzie (1986), we present part of the set-up, equations, and results given in those papers. For a complete panorama and discussion of the assumptions and the model, we refer to those papers.

Take $D \subset \Re^n_+ \times \Re^n_+$ where $n \geq 1$, $u : D \to \Re$, and $\delta \in (0, 1]$. The set $D$ is the *technology*, the function $u$ is the *felicity function* and $\delta$ is the *discount factor*. We say that a sequence $\{x_t\} \subset \Re^n_+$ is a *path* if $(x_t, x_{t+1}) \in D$, for all $t \in N$.[2] We define an *optimal path* from a capital stock $x \in \Re^n_+$, as a path $\{k_t\}$ such that: $k_0 = x$, and

$$\limsup_{T\to\infty} \sum_{t=0}^{T} \left[\delta^{t+1}u(x_t, x_{t+1}) - \delta^{t+1}u(k_t, k_{t+1})\right] \leq 0$$

for all paths $\{x_t\}$, such that $x_0 = x$.

A stationary optimal path $k_t = k$ for all $t \in N$ is called an *optimal steady state* (OSS).

All members of the family of optimal growth problems that we will define later are assumed to satisfy the following assumptions on the technology and felicity functions.

The technology $D$ will be a set in $\Re^{2n}_+$ such that:

**A0** $(0, 0) \in D$

**A1** $D$ is closed and convex, and if $(x, y) \in D$, then for every $(z, w) \in \Re^{2n}_+$ such that $z \geq x$, $0 \leq w \leq y$ we have that $(z, w) \in D$.

**A2** For every $\xi \in \Re_{++}$, there is $\zeta \in \Re_+$, such that: $(x, y) \in D$ and $|x| < \xi$, implies $|y| < \zeta$.

[2] $N = \{0, 1, 2, ..\}$

**A3** There is $(\bar{x}, \bar{y}) \in D$ for which $\bar{y} > \bar{x}$ (existence of an *expansible stock*).

**A4** There are $(M, \gamma) \in \Re^2_{++}$, $\gamma < 1$, such that: $(x, y) \in D$ and $|x| > M$, implies $|y| < \gamma |x|$, (bounded paths).

Throughout the paper the technology $D$ will be fixed and assumed to satisfy A1-A4.

For the felicity function $u : D \to \Re$ we impose:

**A5** $u$ is continuous, concave, and if $(x, y) \in D$, then for every $(z, w) \in \Re^{2n}_+$ such that $z \geq x$, $w \leq y$ we have that $u(z, w) \geq u(x, y)$ (free disposal).

Now for $\delta < 1$ and $(u, D)$ satisfying A1-A5, we define the *value function* $V^{u,\delta}_t(x)$ which values a capital stock at time $t$ by the felicity sums that can be obtained from it in the future:

$$V^{u,\delta}_t(x) := \sup(\lim_{T\to\infty} \sum_{\tau=t+1}^{T} \delta^\tau u(h_{\tau-1}, h_\tau))$$

over all paths $\{h_t\}$ with $h_t = x$. We say that $V^{u,\delta}_t(x)$ is well defined if the sup defined above is finite or $+\infty$. Now we define

$$L := \left\{x \in \Re^n_+ \mid \text{ there exists a path } \{h_t\} \text{ such that } h_0 = x\right\}.$$

Note that $V^{u,\delta}_t(x) \in \Re$ for all $x \in L$ and all $t \in N$.

We will say that prices $\{p_t\}$ *support an optimal path* $\{k_t\}$ if the following conditions are satisfied:

$$\begin{aligned} \delta^{t+1}u(k_t, k_{t+1}) + p_{t+1}k_{t+1} - p_t k_t &\geq \\ \delta^{t+1}u(x, y) + p_{t+1}y - p_t x, \text{ for all } (x, y) &\in D \end{aligned} \tag{1}$$

and

$$V^{u,\delta}_t(k_t) - p_t k_t \geq V^{u,\delta}_t(y) - p_t y, \text{ for all } y \in L, \tag{2}$$

for all $t \in N$.

Prices $\{p_t\}$ satisfying (1) and (2) for all $t \in N$ are called *full Weitzman prices.*

Let $\hat{\delta} \in (0, 1)$ be such that for any $\delta \in \left(\hat{\delta}, 1\right]$, we have $\delta\bar{y} > \bar{x}$. Then , as we will need some further concepts, we recall the following results:

**Lemma 1** *For any* $1 \geq \delta \geq \hat{\delta}$ *and* $(u, D)$ *satisfying A1-A5 there exists* $(k^{u,\delta}, q^{u,\delta}) \in \Re^{2n}_+$, $q^{u,\delta} \neq 0$ *and* $(k^{u,\delta}, k^{u,\delta}) \in D$*, which satisfy:*

$$u(k^{u,\delta}, k^{u,\delta}) \geq u(x, y) \textit{ for all } (x, y) \in D, \textit{ such that } \delta y - x \geq (\delta - 1)k^{u,\delta}$$

*and*

$$\begin{aligned} u(k^{u,\delta}, k^{u,\delta}) + q^{u,\delta}(k^{u,\delta} - \delta^{-1}k^{u,\delta}) &\geq \\ u(x, y) + q^{u,\delta}(y - \delta^{-1}x), \textit{ for all } (x, y) &\in D \end{aligned}$$

*Proof:* See McKenzie (1986), lemmata 6.1 and 7.2. ■

**Theorem 2** *For any* $(u, D)$ *satisfying A1-A6, the stock of capital and the price* $(k^{u,\delta}, q^{u,\delta}) \in \Re^{2n}_+$ *given in Lemma 1, satisfy the following:*

*i) for any* $\delta \in \left(\hat{\delta}, 1\right)$

$$V_t^{u,\delta}(k^{u,\delta}) - \delta^t q^{u,\delta} k^{u,\delta} \geq V_t^{u,\delta}(x) - \delta^t q^{u,\delta} x, \text{ for all } x \in L$$

*for all* $t \in N$.

*ii) for any* $\left(\hat{\delta}, 1\right]$ *the path* $k_t^{u,\delta} = k^{u,\delta}$ *for all* $t \in N$, *is an OSS.*

*iii) for any* $\delta \in \left(\hat{\delta}, 1\right)$ *and for any* $k_0 \in L$ *there exists an optimal path* $\left\{k_t^{u,\delta}\right\}$ *from* $k_0$. *If* $k_0 \in intL$, *any optimal path from* $k_0$ *can be supported by full Weitzman prices* $p_t^{u,\delta} := \delta^t q_t^{u,\delta}$ *in the sense of (1) and (2).*[3]

*Proof:* See McKenzie (1986), theorems 6.1 and 7.1, and a comment on page 1312.■

The point $(k^{u,1}, k^{u,1}) \in D$ is called *the turnpike* in optimal growth theory.

We consider the following assumption:

**A6** $u(x, y)$ is strictly concave at $(k^u, k^u)$.[4]

We will need the definitions that follows, because the concepts of chaos that we will treat in the following section are defined in terms of maps, and therefore, we need a set-up that ensure the existence of a policy function.

First, it is possible to show that there exists a compact, convex set, say $X \subset D$, such that, if $\{h_t\}$ is a path such that $h_0 \in X_1 = L \cap \{x \in \Re_+^n \mid |x| \leq M\}$, then $(h_t, h_{t+1}) \in X$ for all $t \in N$. So, in order to study the long run behavior of the system, without loss of generality, we will restrict the analysis to the set $X$, because any optimal path will enter, sooner or later, in that set. For details on these sets, see Guerrero-Luchtenberg (2000).

Now, let denote by $\hat{L}$ the set of points $k_0 \in L$ for which there exist full Weitzman prices for a given felicity function $\tilde{u}$ . We recall that, given A0 and A5, full Weitzman prices exist for any $k_0 \in \Re_+^n \backslash \{0\}$ , so $\hat{L} = \Re_+^n \backslash \{0\}$ (see Guerrero-Luchtenberg (2000)).

Second, we still assume

**A7** for a given $u$ satisfying A0-A6 any optimal path from $k_0 \in \hat{L}$ has bounded supporting prices for $\delta \in \left(\hat{\delta}, 1\right)$ in the sense that if $\{k_t\}$ is any optimal path from $k_0$ and $\left\{\delta^t q_t^{u,\delta}\right\}$ is a full Weitzman supporting prices, then $\sup_{\delta \in (\hat{\delta}, 1]; t \geq 0} \left\{\left|q_t^{u,\delta}\right|\right\} < \infty$.

and

**A8** any $x \in L$ is the initial capital stock of a unique optimal path.

Henceforth we will denote the set $Proj_1 X$ by $K$. Under A0-A8 it is possible to

[3] We will use the notation $\left\{k_t^{u,\delta}\right\}$ for an optimal path that is not an OSS and $\left\{\delta^t q_t^{u,\delta}\right\}$ for the corresponding Weitzman supporting prices, whereas $\{k^{u,\delta}\}$ and $\{\delta^t q^{u,\delta}\}$, when we refer to the OSS.

[4] We say that a concave function $u : D \rightarrow \Re$ is strictly concave at a point $a$ in $D$ if

$$u(\lambda a + (1 - \lambda)x) > \lambda u(a) + (1 - \lambda)u(x)$$

for all $\lambda \in (0, 1)$ and all $x \in D$ such that $x \neq a$.

show that there is a well defined function $h : K \rightarrow K$, called the policy function, which is continuous and satisfies that $\{k_t\}$ is an optimal path from $x$ if and only if it satisfies that

$$k_{t+1} = h(k_t) \text{ for all } t \in N \text{ with } k_0 = x$$

For a proof see Stokey and Lucas (1989), theorems 4.4, 4.5 and 4.8.

Let then $\{k_t\}$ be an optimal path from $k_0 \in \Re^n_+ \setminus \{0\}$and $\left\{\delta^t q_t^{u,\delta}\right\}$ be the corresponding supporting prices.

Let $Q_u(k_0, \left\{q_t^{u,\delta}\right\}) := \sup_{\delta\in(\hat{\delta},1];t\geq 0} \left\{\left|q^{u,\delta}\right|, \left|q_t^{u,\delta}\right|\right\}$.

Notation and definitions. Take any family as follows,

$$\tilde{U} = \{\tilde{u} : D \rightarrow \Re \mid (\tilde{u}, D) \text{ satisfying A1-A8 }\}$$

and define

$$U(k_0, \left\{\left\{q_t^{\tilde{u},\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\}) = \left\{u : D \rightarrow \Re \mid \ u = \frac{\tilde{u}}{Q_{\tilde{u}}(k_0, \left\{q_t^{\tilde{u},\delta}\right\})},\ \tilde{u} \in \tilde{U}\right\} \quad (3)$$ [5]

**Definition 1** *We will say that any family* $U = U(k_0, \left\{\left\{q_t^{\tilde{u},\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\})$ *as in (3) is relatively compact, if* $\bar{U}$ *is compact relative to the norm of the* sup. [6]

**Definition 2** *We will say that any family* $U = U(k_0, \left\{\left\{q_t^{\tilde{u},\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\})$ *as in (3) satisfies the concavity condition, if for any* $u \in \bar{U}$, *$u$ is strictly concave at* $(k^u, k^u)$.

In the sequel we will consider any family

$$\tilde{U} = \{\tilde{u} : D \rightarrow \Re \mid \ (\tilde{u}, D) \text{ satisfying A0-A8}\} \quad (4)$$

such that for any $k_0 \in \Re^n_+ \setminus \{0\}$ and for any optimal path from $k_0$ with $\left\{\delta^t q_t^{u,\delta}\right\}$ as a corresponding supporting prices for $\delta \in (\hat{\delta}, 1)$, we have that

$$U(k_0, \left\{\left\{q_t^{u,\delta}\right\} \mid \tilde{u}\ \in \tilde{U},\ \delta \in (\hat{\delta}, 1)\right\})$$

satisfies the concavity condition and is relatively compact. In this case, we say that $\tilde{U}$ *satisfies the concavity condition uniformly and that it is uniformly relatively compact.*

All our results are based on the following

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5 Notice that we write $U(k_0, \left\{\left\{q_t^{\tilde{u},\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\})$ in order to emphasize the fact that the family depends on $k_0$ and $\left\{\left\{q_t^{\tilde{u},\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\}$, provided that for a given $\tilde{u} \in \tilde{U}$ and $k_0$, the prices $\left\{q_t^{\tilde{u},\delta}\right\}$ are not necessarily unique.

6 We denote by $\bar{U}$ the closure of $U$ relative to the norm of the sup (the norm of the sup for a given function $u \in U$, is defined as follows, $|u|_\infty = \sup_{x\in X} |u(x)|$).

**Theorem 3** *(A Uniform Neighborhood Turnpike Theorem) Take any $U$ as in (4) that satisfies the concavity condition uniformly and that is uniformly relatively compact. Then, for any $\varepsilon > 0$ there exists $N(\varepsilon)$ and $0 < \delta(\varepsilon) < 1$ such that: For all $\delta(\varepsilon) \leq \delta < 1$ and $u \in U$, we have $\left|k_t^{u,\delta} - k^{u,\delta}\right| \leq \varepsilon$ for all $t > N(\varepsilon)$.*

*Proof:* It follows at once from the theorem 2 in Guerrero-Luchtenberg (2000).
■

***Dynamical systems***

In this section we will give some basic definitions about dynamical systems. Nevertheless, before we start with these definitions, we would like to make one more general comment about the concepts of chaos that we will consider. Most definitions are made for the sake of the study of the long run behavior of the optimal solutions and the chaos then is defined by using concepts entailing, essentially, some kind of uncertainty about final states of the dynamical system under consideration. So the precise formulation of our basic results will be made following this basic idea, that is, the chaos vanishes if the uncertainty vanishes.

A point $y \in \Re^n_+$ is called an *$\omega-$limit point of* $k_0 \in \Re^n_+$ if there is an optimal sequence $\{k_t\}$ from $k_0$ and a subsequence $\{k_{t_s}\}$ of $\{k_t\}$ such that $\lim_{s\to\infty} k_{t_s} = y$. Denote by $W(k_0)$ the set of all *$\omega-$limit* points of $k_0$, called the *$\omega-$limit set of* $k_0$.

A point $y \in \Re^n_+$ is called *recurrent* if $y \in W(y)$. We will denote by $R$ the set of all recurrent points, that is

$$R := \left\{y \in \Re^n_+ \mid y \in W(y)\right\}.$$

A *discrete dynamical system* can be defined by a pair $(K, h)$ where $K \subset \Re^n$ and $h$ is a function from $K$ to $K$. The set $K$ is called the *state space.*

For any $x \in K$, define $h^0(x) = x$, and for any $k \geq 1$ ($k \in N$)

$$h^k(x) = h(h^{k-1}(x)).$$

We say that the sequence $\{h^t(x)\}$ *is generated by the iterations of $h$ from $x$*, and that $h^k$ *is the iteration of $h$ up to order $k$.* Also, the sequence $\{h^t(x)\}$ is called the *orbit from $x$.* A point $x \in K$ is called *a periodic point* of $h$, if $\{h^t(x)\}$ is finite and $h^p(x) = x$ for some $p > 1$. The smallest such $p$ is called *the period of $x$.* If there exists a periodic point of period $k$, then we say that the dynamical system $(K, h)$ has *period-k cycles.*

For the sake of the exposition, we will prove the following theorem, which will simplifies notably the proofs of the main results of this paper.

For any $\delta \in \left(\hat{\delta}, 1\right)$ (recall that $\hat{\delta}$ was defined in Lemma 1) and $u \in \tilde{U}$, let $W^{u,\delta}(k_0)$ denote the $\omega$-limit set of $k_0$. Take any family $\tilde{U}$ and define the function $f_{\tilde{U}}$ : $\left(\hat{\delta}, 1\right) \rightarrow \Re \cup \{\infty\}$ given by

$$f_{\tilde{U}}(\delta) = \sup_{k_0 \in \hat{L};\, u \in \tilde{U}} \left\{ \sup_{y \in W^{u,\delta}(k_0)} \left|y - k^{u,\delta}\right| \right\}$$

then, we can prove the following:

**Theorem 4** *Take any $U$ as in (4) that satisfies the concavity condition uniformly and that is uniformly relatively compact. Then,*

$$\lim_{\delta \to 1} f_U(\delta) = 0$$

*Proof:* The proof is by contradiction. Suppose the theorem is false, then there exists a family $U$ satisfying the conditions of the theorem, such that $\lim_{\delta \to 1} f_U(\delta) \neq 0$. Then there is an $\varepsilon > 0$, a sequence $\{\delta_l\} \subset \left(\hat{\delta}, 1\right)$ such that $\delta_l \to 1$, and a sequence $k_0^l \in \Re_+^n \setminus \{0\}$ such that there exists $y_l \in W^{u_l,\delta_l}(k_0^l)$ such that

$$\left|y_l - k^{u_l,\delta_l}\right| > \varepsilon \text{ for all } l \text{ large enough}$$

Now, by construction of the set $K$, for any $k_0^l \in \Re_+^n \setminus \{0\}$ , if $\{k_t(k_0^l)\}$ denotes the optimal path from $k_0^l$, then there exists an integer $t_l$ such that $k_{t_l}(k_0^l)$ will belong to $K$.

Let $x_0^l \in K$ denote such a point, that is, we write

$$x_0^l := k_{t_l}(k_0^l) \text{ for all } l \in N.$$

Notice that from the definition of $\omega-$limit sets, one can prove that

$$W^{u_l,\delta_l}(x_0^l) = W^{u_l,\delta_l}(k_0^l)$$

Therefore, for all $l \in N$ we have that $y_l \in W^{u_l,\delta_l}(k_0^l)$, implies

$$y_l \in W^{u_l,\delta_l}(x_0^l)$$

then there exists a sequence $\left\{k_t^{u_l,\delta_l}\right\}$ from $x_0^l$ and a sub-sequence $\left\{k_{t_k}^{u_l,\delta_l}\right\} \subset \left\{k_t^{u_l,\delta_l}\right\}$ such that

$$\lim_{k \to \infty} k_{t_k}^{u_l,\delta_l} = y_l.$$

Therefore, $\left|k_{t_k}^{u_l,\delta_l} - k^{u_l,\delta_l}\right| > \varepsilon$ for all $k$ and $l$ large enough, a contradiction with the theorem 2. This completes the proof of the theorem 3.■

### *The main results*

First, we consider the concept of topological chaos.

#### *Topological chaos*

We will say that a dynamical system $(K, h)$ displays *topological chaos* or that is *topologically chaotic* if there exists a subset $\Sigma \subset K$ such that:

**T1** $\Sigma$ is uncountable

**T2** $\Sigma$ does not contain any periodic point of $h$

**T3** For any $(x, y) \in \Sigma \times \Sigma$ such that $x \neq y$,

$$\liminf_{t\to\infty} \left|h^t(x) - h^t(y)\right| = 0 \text{ and } \limsup_{t\to\infty} \left|h^t(x) - h^t(y)\right| > 0$$

**T4** For any periodic point $y \in K$ and any $x \in \Sigma$,

$$\limsup_{t\to\infty} \left|h^t(x) - h^t(y)\right| > 0.$$

A set $\Sigma \subset K$ is called *a scrambled set* if it satisfies T1-T4.

Also, we say that *an optimal growth problem* $(u, D, \delta)$ *displays topological chaos* if the dynamical system $(h, K)$ displays topological chaos, where $h$ is the policy function of $(u, D, \delta)$.

Note that T3 and T4 are indeed a way to describe some type of uncertainty about final states, because it may be possible that two optimal paths from different points may not converge to the same point; further, the may not even converge to a same periodic point; also, no periodic point can be globally stable, again, a very undesirable fact regarding final states. Nevertheless, the relevance of this type of chaos depends on how "big" is the scrambled set in terms of probabilistic concepts. Indeed, it has been proven that the scrambled set may have zero Lebesgue measure (see Collet and Eckmann (1986)), in which case there is zero probability of choosing points satisfying T3 or T4. Notice that this may not imply that there is zero probability of observing topological chaos. Think of the case when the scrambled set is a global attractor.

Now we will show that the uncertainty implied by topological chaos vanishes as the discount factor tends to one. The intuition of the result is the following. As we have commented in the paragraph above, the concept of topological chaos entails the impossibility of certain predictions in the long run. Therefore, if we have a family such that for any value of the discount factor there is a member of the family such that the corresponding optimal growth problem displays topological chaos, the expression 'chaos vanishes as the discount factor tends to one,' means that if the discount factor is large enough, the distance from any two possible final states is very close to zero, and then no uncertainty in the long run would be relevant. Formally:

**Theorem 5** *Take any $U$ as in (4) that satisfies the concavity condition uniformly and that is uniformly relatively compact. Suppose that for every $\delta \in (0,1)$ there exists $u_\delta \in U$ such that the optimal growth problem $(u_\delta, D, \delta)$ displays topological chaos. Let $h^{u_\delta}$ denote the policy function of optimal growth problem $(u_\delta, D, \delta)$. Let $\Sigma^{h^u}$ denote the scrambled set of $(h^{u_\delta}, K)$. Let*

$$C_{T3}(\delta) = \sup_{(x,y)\in\Sigma^{h^{u_\delta}}\times\Sigma^{h^{u_\delta}} \text{ such that } x\neq y} \{\limsup_{t\to\infty} |(h^{u_\delta})^t(x) - (h^{u_\delta})^t(y)|\} \text{ and}$$

*$C_{T4}(\delta)$ given by*

$$\sup_{y\in K \text{ piriodic point of } h^{u_\delta} \text{ and any } x\in\Sigma^{h^{u_\delta}}} \{\limsup_{t\to\infty} |(h^{u_\delta})^t(x) - (h^{u_\delta})^t(y)|\}.$$

*Then we have*

$$\lim_{\delta\to 1} C_{T3}(\delta) = 0 \tag{5}$$

*and*

$$\lim_{\delta \to 1} C_{T4}(\delta) = 0 \tag{6}$$

*Proof*: Suppose the corollary is false. Then either (5) or (6) is false. Suppose then that (5) is false. The case when (6) is false is analogous.

If (5) is false for some family $U$ satisfying the conditions of the theorem, then there is an $\varepsilon > 0$, a sequence $\{\delta_l\} \subset (0,1)$, and a sequence $\{u_l\} \subset U$ such that $\delta_l \to 1$, and such that there exist points $(x_l, y_l) \in \Sigma^{h^l} \times \Sigma^{h^l}$ such that $x_l \neq y_l$ satisfying

$$\limsup_{t \to \infty} \left|(h^l)^t(x_l) - (h^l)^t(y_l)\right| > \varepsilon. \tag{7}$$

for all $l$ large enough, where $h^l$ denotes the policy function of the optimal growth problem $(u^l, D, \delta_l)$ and $\Sigma^{h^l}$ its corresponding scrambled set.

Note that (7) implies that there exist points $(z_1^l, z_2^l) \in W(x_l) \times W(y_l)$ (where $W(x_l)$ denotes the $\omega-$limit set of $x_l$ of the dynamical system $(h^l, K)$) such that

$$\left|z_1^l - z_2^l\right| > \varepsilon \tag{8}$$

for all $l$ large enough, a contradiction with Theorem 3, because (8) implies that

$$\liminf_{l \to \infty} f_U(\delta_l) \geq \frac{1}{2}\varepsilon.$$

This completes the proof of the theorem 4 ■

Now we consider the concept of sensitive dependence on initial conditions.

*Sensitive dependence on initial conditions*

We say that a dynamical system $(K, h)$ has, for a given $\varepsilon > 0$, *$\varepsilon-$sensitivity* or that it displays *$\varepsilon-$sensitive dependence on initial conditions* if there is a set $E \subset K$ of strictly positive Lebesgue measure such that, for every $y \in E$ and every neighborhood $B$ of $y$, there exists a $z \in B$ and a $t \in N$ such that

$$\left|h^t(y) - h^t(z)\right| > \varepsilon.$$

Intuitively, the general idea of this definition is that if $y \in E$, no matter how close to $y$ we are studying the behavior of the system, there exists a point $z$ that will be $\varepsilon-$separated sooner or later. So, if for some reason we are not able to distinguish between two points that are not $\varepsilon-$separated at the beginning of period of consideration, we will be able to distinguish between them after some time. Clearly, this is a way to describe some undesirable behavior of a dynamical system regarding final states, in the sense that minor changes on the initial conditions result, probably, in significant differences after some time.

We will say that *an optimal growth problem* $(u, D, \delta)$ *displays* $\varepsilon - sensitivity$ if the dynamical system $(h, K)$ displays $\varepsilon - sensitivity$, where $h$ is the policy function of $(u, D, \delta)$.

Now we will prove that if there is family $U$ as in (4) that satisfies the concavity condition uniformly and that is uniformly relatively compact, such that for every $\delta \in (0,1)$ there is a member $u_\delta$ of $U$ and an $\varepsilon_{u_\delta} > 0$ such that $(u, D, \delta)$ displays

$\varepsilon_{u_\delta}$−sensitive dependence on initial conditions, then for avery $\varepsilon > 0$, there is number $\delta(\varepsilon)$, such that if $\delta$ is larger than $\delta(\varepsilon)$, we have that $\varepsilon_{u_\delta}$ cannot be larger than $\varepsilon$. This precisely means that the $\varepsilon_{u_\delta}$−sensitive dependence on initial conditions is no longer relevant for $\delta$ large enough. Formally:

**Theorem 6** *Take any $U$ as in (4) that satisfies the concavity condition uniformly and that is uniformly relatively compact, such that for every $\delta \in (0,1)$ there exists a member $u_\delta \in U$ and an $\varepsilon_\delta > 0$ such that the optimal growth problem $(u_\delta, D, \delta)$ displays $\varepsilon_\delta$-sensitivity, then*

$$\lim_{\delta \to 1} \varepsilon_\delta = 0$$

*Proof*: Suppose that the corollary is false. In this case there is a family $U$ , an $\varepsilon > 0$ , a sequence $\{\delta_l\} \subset (0,1)$ , a sequence $\{u_l\} \subset U'$ and a sequence $\{\varepsilon_l\} \subset \Re_+$ such that $\delta_l \to 1$, $\varepsilon_l \geq \varepsilon$ for all $l$ large enough and for all $l$ the dynamical system $(h^l, K)$ displays $\varepsilon_l$−sensitivity, where $h^l$ denotes the policy function of the optimal growth problem $(u^l, D, \delta_l)$.

Now, notice that if $K$ is compact and $h$ is continuous, then if $(K,h)$ has $\varepsilon$*−sensitivity*, then for any $l \in N$ there exist $y_l \in E\setminus\{0\}$, $z_l \in K\setminus\{0\}$ and $t_l \in N$ with $t_l > l$, such that

$$\left|h^{t_l}(y_l) - h^{t_l}(z_l)\right| \geq \varepsilon$$

because given $\varepsilon > 0$, for any $l \in N$ fixed, there exists a $\alpha > 0$ such that

$$(y,z) \in K \times K \text{ and } |y - z| < \alpha, \text{ implies } \left|h^t(y) - h^t(z)\right| \leq \varepsilon \text{ for all } t \leq l$$

Consequently, for every $l \in N$ there exist $y_l \in E\setminus\{0\}$, $z_l \in K\setminus\{0\}$ and $t_l \in N$ with $t_l > l$, such that

$$\left|(h^l)^{t_l}(y_l) - (h^l)^{t_l}(z_l)\right| \geq \varepsilon_l \geq \varepsilon$$

Therefore,

$$\liminf_{l \to \infty} \left|(h^l)^{t_l}(y_l) - (h^l)^{t_l}(z_l)\right| \geq \varepsilon$$

and thus

$$\liminf_{l \to \infty} f_U(\delta_l) \geq \varepsilon$$

a contradiction with the theorem 3. Then the theorem 5 is proven. ■

*The general statement 'chaos vanishes as the discount factor tends to one'*

As we have commented before, we consider that a dynamical system displays some kind of chaos, if there is some 'deterministic uncertainty,' about the possible final states of the system, in such a way that no predictions in the long run are trustable. So our result here, then, express the idea that, even if the system displays some kind of chaos, if the 'deterministic uncertainty,' about the possible final states of the system is negligible, in the sense that the distance between two possible final states remain lower than any $\varepsilon$ small enough, given that the discount factor is large enough, then the chaos is not relevant.

Formally:

**Theorem 7** *Take any $U$ as in (4) that satisfies the concavity condition uniformly and that is uniformly relatively compact. Take, for a given $u \in U$ and $\delta$, any two points $x$ and $y$ in $L$ and the corresponding $w-$limit sets, $W_{u,\delta}(x)$ and $W_{u,\delta}(y)$. Now take $d(W_{u,\delta}(x), W_{u,\delta}(y)) = \sup \{|z - v| \mid z \in W_{u,\delta}(x), v \in W_{u,\delta}(y)\}$ (the maximum distance between any two possible $w-$limit points of $x$ and $y$ respectively). Define now $D(u,\delta) = \sup \{d(W_{u,\delta}(x), W_{u,\delta}(y)) \mid (x,y) \in L^2\}$ (the maximum distance between any two possible $w-$limit points of the system $(u, D, \delta)$) and*
*$D_U(\delta) = \sup \{D(u,\delta) \mid u \in U\}$ (the maximum distance between any two possible $w-$limit points of the family of systems). Then*

$$\lim_{\delta \to 1} D_U(\delta) = 0$$

*Proof:* Take any pair $(u,\delta)$. Now, take any two possible initial states $(x,y) \in L^2$, and any two possible $w-$limit points of $x$ and $y$ respectively, say $z(x) \in W_{u,\delta}(x)$ and $v(y) \in W_{u,\delta}(y)$. Recall that for $\delta > \hat{\delta}$, $k^{u,\delta}$ is well defined (the lemma 1 and the theorem 1). Hence, suppose that $\delta > \hat{\delta}$. Now we have $|z(x) - v(y)| \le |z(x) - k^{u,\delta}| + |v(y) - k^{u,\delta}|$. Consequently, $D(u,\delta) \le \sup_{y \in W_{u,\delta}(x)} |y - k^{u,\delta}| + \sup_{y \in W_{u,\delta}(y)} |y - k^{u,\delta}|$. Therefore, $D_U(\delta) \le f_U(\delta) + f_U(\delta) = \frac{1}{2} f_U(\delta)$. Hence, as a direct consequence of theorem 3, $\lim_{\delta \to 1} D_U(\delta) = 0$, and this finishes the proof of the theorem 6.

## *Conclusions*

As commented in the introduction, in order to justify what we called ' the uniform comparative analysis' used in empirical works, neither the standard turnpike theorems nor the specific lower bound fund in relation to some fixed type of chaos can be cited. In this paper, we show how the theorem 2 in Guerrero-Luchtenberg (2000) can be applied in order to rule out the topological chaos and the $\varepsilon-$sensitive dependence on the initial conditions, two concepts not analyzed in previous studies. Furthermore, and more important, we prove the theorem 6, in which no special concept of chaos is assumed, and therefore that theorem 6 is an appropriate justification for the uniform comparative analysis. It rest then to study if another result of this type is possible under weaker conditions. Also, it would be interesting to find the necessary conditions for the result in the theorem 6 to hold. Both questions are left for future research.

## *References*

M. Boldrin and L. Montrucchio, On the Indeterminacy of Capital Accumulation Paths, *Journal of Economic Theory* **40**, 26-39 (1986).

P. Collet and J. P. Eckmann, Iterated Maps on the Interval as Dynamical Systems, Progress in Physics, (1986).

C. Guerrero-Luchtenberg, 'A Uniform Neighborhood Turnpike Theorem and Applications,' *Journal of Mathematical Economics,* **34** (2000), 329-357.

T. Mitra, An Exact Discount Factor Restriction for Period-Three Cycles in Dynamic Optimization Models, *Journal of Economic Theory,* **69** (1996), 281-305.

T. Mitra, On the Relationship between Discounting and Complicated Behavior in Dynamic Optimization Models, *Journal of Economic Behavior and Organization,* **33(3-4)**, Special Issue January 1998, 421-34..

L. W. McKenzie, Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics, *Handbook of Mathematical Economics,* Vol. III, Chapter 26 (1986), 1281-1355.

L. Montrucchio and G. Sorger, Topological Entropy of Policy Functions in Concave Dynamic Optimization Models, *Journal of Mathematical Economics,* **25** (1996), 181-194.

K. Nishimura , G. Sorger and M. Yano, Ergodic Chaos in Optimal Growth Models with Low Discount Rates, Symposium on Dynamical systems, *Economic Theory,* **4** (1994), 705-717.

K. Nishimura and M. Yano, Nonlinear Dynamics and Chaos in Optimal Growth: an Example, *Econometrica,* Vol. 63, **4** (1995), 981-1001.

K. Nishimura and M. Yano, On the Least Upper Bound of Discount Factors that Are Compatible with Optimal Period-Three Cycles, *Journal of Economic Theory,* **69** (1996), 306-333.

J. A. Scheinkman, On Optimal Steady States of $n$-sector Growth Models when Utility is Discounted, *Journal of Economic Theory,* **12** (1976), 11-30.

G. Sorger, Period Three Implies Heavy Discounting, *Mathematics of Operations Research,* **19** (1994), 1007-1022.

N. L. Stokey and R. Lucas (with E. Prescott) , Recursive Methods in Economics Dynamics, Harvard University Press, Cambridge, MA, 1989.