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The Selection of Preferences in OLG Models with Endogenous Heterogeneity

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Abstract

We present a model for the selection of preferences in an Overlaping Generations setting with heterogeneous agents. The distribution of agents' types evolves according to a SPS dynamics {Sign-Preserving Selection dynamics} where types with higher consumption levels thrive at the expense of others, thus yielding endogenous heterogeneity. The only long-run steady states entail an interest rate equal to the population growth rate (Golden Rule), and are hence Pareto optimal. Suboptimal no-trade equilibria are not steady states under endogenous heterogeneity., and therefore allow this program to fulfill its objective which is to guaranty workers pensions.

Resumen

Presentamos un modelo de selección de preferencias en un entorno de generaciones solapadas con agentes heterogéneos. La distribución de tipos evoluciona de acuerdo a una dinámica del tipo SPS (Sign-Preserving Selection Dynamics) según la cual los tipos con mayor consumo crecen a expensas de los de menor consumo, y de esta manera se obtiene una distribución de tipos heterogénea. Los únicos estados estacionarios son aquellos en los cuales el tipo de interés coincide con la tasa de crecimiento de la población (la regla de oro), y por lo tanto son óptimos en sentido de Pareto. Los estados estacionarios correspondientes a la situación en la que no hay intercambio desaparecen cuando se tiene heterogeneidad.

Introduction

This paper presents a model for the *selection of preferences* by endogenizing the proportion of agents endowed with different preferences in a standard Overlapping Generations model.

The selection or evolution of preferences is a fundamental issue for economic theory (see e.g. L. Samuelson (2001)). Most of the current literature refers to game-theoretic settings and is closely related to the field of evolutionary game theory (see e.g. Friedman (1991) or Weibull (1995)). Quite naturally, this literature is based on the darwinian idea that preferences leading to a higher success (or *fitness)* will spread in the population, at the expense of others. The *indirect evolutionary approach* (see Güth and Yaari (1992) and Güth (1995)) distinguishes between subjective payoffs (representing preferences) and objective ones (representing fitness). The latter are then used as a measure of the evolutionary success of the former.

Rather than introducing fitness, we postulate consumption levels as the appropriate measure of success. The idea is as follows. Agents take life-long consumption decisions rationally, optimizing their preferences. Those very preferences, though, are acquired through some learning mechanism. In game-theoretic settings, it is known that considerations of e.g. imitation of succesful agents lead to dynamic behavior which can be described by evolutionary dynamics (see Weibull (1995)). We base "success" not on utilities, which are neither observable nor comparable, but on consumption, which is both, observable and comparable. A rationale for this could be that agents are accultured (i.e., acquire their preferences) in their childhood on the basis of visibly successful role models.

We study a family of (discrete-time) dynamics such that the preferences of agents who on average (old and young agents) consume more than others spread in the population. These are the analogue of the *Sign-Preserving Selection* (SPS) dynamics of evolutionary game theory (see Ritzberger and Weibull (1995)). Our results do not depend on the particular dynamics assumed, as long as it is of the SPS type.

Our model can be viewed as an attempt to endogenize heterogeneity in an Overlaping Generations setting. The research agenda would be to compare the results from the standard model with those for the model proposed here. Therefore, as a benchmark, we first study the model where each generation is a scaled replica of the previous one, i.e. heterogeneity is exogenously fixed. It is known that, in this case, there are two types of steady states. In the first, called *Golden Rule* steady states, the (gross) interest rate equals the population growth rate (the *biological interest rate*) (see P. Samuelson (1958) and Starrett (1972)). These states are Pareto optimal. In the second, called *no-trade* steady states, there is no trade between young and old agents (a situation sometimes referred to as the Garden of Eden case). These states are suboptimal in general.

A second well-known fact about the model without endogenous heterogeneity is the so-called Samuelson's Impossibility Theorem (see P. Samuelson (1958) and Gale (1973)). Stated in words, this result says that, if an economy starts with only young agents (or only old agents), it can never converge to the optimal, Golden Rule steady state, unless, by chance, the exogenous population profile is such that the Golden Rule state is also a no-trade state.

In sharp constrast to the previous results, in the model with endogenous heterogeneity, the no-trade steady states disappear. In other words, once the system is completed with an additional dimension corresponding to the selection of preferences, in those states the population distribution of types does not remain steady. In the interior of the state space, (except for degenerate cases) only the optimal Golden Rule states remain.

An immediate implication is that, if an equilibrium path converges to an interior (i.e. with heterogeneity) steady state, the latter must be of the Golden Rule type, and hence Pareto optimal. This overcomes one of the most uncomfortable properties of the basic Overlapping Generations model, namely the possibility of convergence to suboptimal steady states.

A second implication is the failure of Samuelson's Impossibility Theorem in our framework. With endogenous heterogeneity, if an economy starts in the Garden of Eden case (e.g. with only young agents), convergence to an interior point can only happen if the population proportions adjust in precisely such a way that the system reaches a very particular distribution where the associated no-trade state coincides with the Golden Rule steady state and is, hence, optimal.

In its basic motivation, our paper is related to the mentioned literature on the evolution of preferences in game-theoretic settings (see Section 3 for further discussion). We would like to stress also the conceptual relation with the broad literature of learning in Overlapping Generations models. In that literature, agents' expectations are explicitly taken into account to give rise to a family of *adaptive learning* dynamics (see Lucas (1986)). Part of the aim of such models is to discriminate between autarkic, no-trade steady states and monetary (Golden rule) states, hopefully selecting the latter as the outcome of a learning process (see e.g. Duffy (1994) and the references therein).

The present paper is, to our knowledge, the first to study endogenous heterogeneity of preferences in an Overlapping Generations setting. There are, in general, few studies on the general issue of endogenous heterogeneity. A totally different approach is that of Aiyagari (1989), who shows the existence of equilibria in which some sequences of agents may be linked by positive bequests (and therefore act as infinitely lived agents), while others are not. That model can be though of as a endogenous mix of finitely and infinitely lived agents. The properties of these equilibria, though, cannot be compared with those of equilibria in the standard model, without endogenous heterogeneity.

The paper is organized as follows. Section 2 describes the Overlapping Generations economy with heterogeneous agents and characterizes the associated steady states. We introduce an extended example where agents are endowed with (arbitrarily many) different patience levels. Section 3 introduces the evolutionary dynamics on agents' types and analyzes the corresponding steady states. We return then to the example with different patience levels to better illustrate our findings. Section 4 presents an example of SPS dynamics and reports on some simulations for illustrative purposes. Section 5 concludes.

OLG with exogenous heterogeneity

The model

We consider a double-ended, stationary overlapping-generations economy with no production where agents live for two periods. There is a single, perishable good each period. At each $t \in \mathbb{Z}$, a generation of agents of size $N_t > 0$ is born. Agents belong to one of finitely many different types $h \in H$. The proportion of agents of type $h \in H$ born at t is denoted by ρ_t^h, so that $N_t^h = \rho_t^h \cdot N_t$ is the size of the subpopulation of agents of type h born at t. The population profile at time t is given by a vector $\boldsymbol{\rho}_t = \left(\rho_t^h\right)_{h \in H}$, that is, the distribution of types in the population at t.

The first period of their life, agents receive a fixed endowment, which we normalize to 1. The second period, endowment is denoted by $e > 0$. Agents of type $h \in H$ have an intertemporal utility function

$$u^h(c_0^h, c_1^h)$$

strictly increasing in both arguments, where c_0^h, c_1^h are the consumption levels in the first and second period of their lives, respectively.

An economy so described is then given by $\mathcal{E} = \left(e, \left\{u^h\right\}_{h \in H}, (N_t, \boldsymbol{\rho}_t)_{t \in \mathbb{Z}}\right)$.

Definition 1 *An* equilibrium *for the economy $\mathcal{E}$ is a pair* $(\mathbf{C}, \mathbf{R})$ *where* $\mathbf{C} = \left\{\left(c_0^h(t), c_1^h(t+1)\right)\right\}_{h \in H}$ *is a sequence of agents' consumption plans and* $\mathbf{R} = \{R_t\}_{t \in \mathbb{Z}}$ *is a sequence of strictly positive quantities (called interest rates), such that*

(i) For all $h \in H$ and all $t \in \mathbb{Z}$, $\left(c_0^h(t), c_1^h(t+1)\right)$ solves

$$\left.\begin{array}{ll} \max & u^h\left(c_0^h, c_1^h\right) \\ \text{s.t.} & R_t c_0^h + c_1^h = R_t + e \\ & c_0^h, c_1^h \geq 0 \end{array}\right\} \qquad (1)$$

and

(ii) market clearing holds for all $t \in \mathbb{Z}$:

$$\sum_{h \in H} \left(N_t^h c_0^h(t) + N_{t-1}^h c_1^h(t)\right) = N_t + e N_{t-1}. \qquad (2)$$

where $N_t^h = \rho_t^h \cdot N_t$.

An equilibrium is a steady state *if* $\left(c_0^h(t), c_1^h(t+1)\right) = \left(c_0^h, c_1^h\right)$ *for all t and all* $h \in H$.

The quantity $R_t = 1 + r_t$ is interpreted as the gross interest rate (yield) from period t to period $t+1$. That is, the price of the good in period $t+1$ measured in units of the good in period t is $\frac{1}{R_t}$. Agents' consumption decisions imply then a decision about whether to become a borrower or a lender.[1]

[1] Strictly speaking, intergenerational lending would require the intermediation of a financial institution with an infinite lifespan.

Given an equilibrium, we define

$$A_t^h = 1 - c_0^h(t)$$

and

$$A_t = \frac{1}{N_t} \sum_{h \in H} N_t^h A_t^h = \sum_{h \in H} \rho_t^h A_t^h = 1 - \sum_{h \in H} \rho_t^h c_0^h(t).$$

We call the quantity A_t the average or *aggregate savings* of the economy at time t. This quantity plays a fundamental role in our analysis. If $A_t > 0$, young agents born at t are saving in the aggregate, while if $A_t < 0$, they are in debt. If $A_t = 0$, there is no trade between young and old agents at t. Gale (1973)calls A_t (without averaging) *aggregate assets* of the economy, since it could be defined more generally for a model with production by including firms' capital.

The next proposition will allow us to classify the possible equilibria of a given economy according to the associated aggregate savings. Furthermore, it will greatly simplify the computation of steady state equilibria in the sequel.

Proposition 1 *Let $\mathcal{E}$ be an economy as above. Then, in any equilibrium,*

$$A_t = \left(\frac{N_{t-1}}{N_t}\right) R_{t-1} A_{t-1} \tag{3}$$

for all t.

Proof. Dividing (2) by N_{t-1}, we obtain

$$\sum_{h \in H} \left(\frac{N_t}{N_{t-1}} \cdot \rho_t^h c_0^h(t) + \rho_{t-1}^h c_1^h(t) \right) = \frac{N_t}{N_{t-1}} + e.$$

From the budget constraint of problem (1) at time $t-1$, we have

$$e - c_1^h(t) = -R_{t-1} A_{t-1}^h$$

and hence (2) becomes

$$\begin{aligned} 0 &= \frac{N_t}{N_{t-1}} + e - \sum_{h \in H} \left(\frac{N_t}{N_{t-1}} \cdot \rho_t^h c_0^h(t) + \rho_{t-1}^h c_1^h(t) \right) = \\ &= \frac{N_t}{N_{t-1}} \left(1 - \sum_{h \in H} \rho_t^h c_0^h(t) \right) + \left(e - \sum_{h \in H} \rho_{t-1}^h c_1^h(t) \right) = \\ &= \frac{N_t}{N_{t-1}} A_t + \sum_{h \in H} \rho_{t-1}^h \left(e - c_1^h(t) \right) = \frac{N_t}{N_{t-1}} A_t - R_{t-1} A_{t-1} \end{aligned}$$

as stated. ■

We proceed now to give a classification of equilibria. Consider an equilibrium $(\mathbf{C}, \mathbf{R})$. Since $R_t > 0$ for all t, it follows from (3) that the sign of A_t is constant in t. If $A_t = 0$, we say that $(\mathbf{C}, \mathbf{R})$ is a *Garden of Eden equilibrium*. The name comes from the fact that, in this case, each period young agents consume their endowments (in the aggregate), so that there is no trade with the old agents (from the young agents' point of view, it is as if they were alone in the economy). Note that, although in a Garden of Eden equilibrium there is no intergenerational trade, there will in general be intragenerational trade, since agents are not identical.

If $A_t < 0$, we say that $(\mathbf{C}, \mathbf{R})$ is a *Classical Case equilibrium*, where young agents are in debt in the aggregate, so that the economy systematically borrows from the future. The name "classical" comes from Fisher's informal explanation for positive (net) interest rates (see Fisher (1961)), which proceeds (roughly) as follows. Agents desire a higher consumption in their youth than their income would allow for. Therefore, they must transfer income from their latter years, and hence relative prices must fall as time goes on, i.e. net interest rates are positive. This implication, though, can not be drawn in the current model (see P. Samuelson (1958)).

If $A_t > 0$, we say that $(\mathbf{C}, \mathbf{R})$ is a *Samuelson Case equilibrium*, where young agents are lending in the aggregate, so that the economy systematically saves. This is the case defended by P. Samuelson (1958) as being more realistic, while not incompatible with positive interest rates.[2]

The discussion above gives a complete classification of equilibria. That is, given an equilibrium, the aggregate savings are either always zero (Garden of Eden case), always strictly negative (Classical case), or always strictly positive (Samuelson case).

We want to remark that we could allow for zero gross interest rates in the definition of equilibrium. In this case, though, 3 would imply that any such equilibrium would correspong to the Garden of Eden case.

Economies with constant population growth

We now particularize the general framework of the previous subsection to the case of constant population growth, since this is the case that has received more attention in the literature. We consider now the benchmark case in which each generation is a proportional replica of the previous one, that is, the population profile is constant. We will maintain the constant population growth assumption throughout, but, later on, we will endogenize the evolution of the population profile.

Definition 2 *An economy* $\mathcal{E} = \left(e, \{u^h\}_{h \in H}, (N_t, \boldsymbol{\rho}_t)_{t \in \mathbb{Z}}\right)$ *has a* constant (gross) population growth rate γ *if* $N_{t+1} = \gamma N_t > 0 \; \forall \, t$, *and it has* constant proportions *if* $\boldsymbol{\rho}_t = \boldsymbol{\rho} \; \forall \, t$.

Applying (3), we obtain:

Corollary 2 *Let $\mathcal{E}$ be an economy with a constant population growth rate γ. Then, in any equilibrium,*

$$A_t = \frac{1}{\gamma} R_{t-1} A_{t-1} \tag{4}$$

for all t. Moreover, if, in addition, $\mathcal{E}$ has constant proportions, then in any steady state either $R_t = \gamma \;\; \forall \, t$ or $A_t = 0 \;\; \forall \, t$.

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[2] Samuelson's analysis gave rise to a long discussion in the literature, concerning both the realism of his assumptions and the optimality of the resulting equilibria. See Cass and Yaari (1966), Lerner (1959 (a) and (b), Meckling (1960 (a) and (b), Phelps (1961), P. Samuelson (1958 (a) and (b), and Starrett (1973).

Hence, we re-obtain the well-known result that, in an OLG economy with constant proportions and constant population growth rate, there are two types of steady states. In the first, the (gross) interest rate equals the population growth rate (the *biological interest rate*) (see Samuelson (1958) and Starret (1972)). These are called *Golden Rule* steady states. It is a standard result in the literature that these steady states are Pareto optimal.[3] The second type of steady states correspond to Garden of Eden equilibria where there is no trade between young and old agents. We refer to these as the *no-trade* steady states (also called non-monetary balanced steady states or inside-money equilibria), which are non-optimal in general.[4] The name no-trade is adopted for brevity, but it would be more appropriate to call them no-intergenerational-trade states, since in these states there might be intragenerational trade.

In our case, the sets of equilibria are manifolds parameterized by the agents' proportions given by the vector ρ. Balasko and Lang (1998) study the structure of the equilibrium sets as manifolds parameterized by the agents' endowments.

Suppose that, instead of a double-ended OLG economy, we would consider a one-ended economy, that is, time starts at $t = 0$. If there are no old agents at $t = 0$, the economy is, by necessity, in the Garden of Eden case (and this is the origin of the name). Suppose that the Golden Rule steady state for this economy entails $A_t > 0$ (and constant). Since the economy starts in the Garden of Eden case, with $A_t = 0$, it follows that convergence to the Golden Rule is impossible. This fact, first conjectured by P. Samuelson (1958), was proved in Gale (1973) and is referred to as Samuelson's Impossibility Theorem. In words, if an economy starts with only young agents (or only old agents), it can never converge to the optimal, Golden Rule steady state, except for very particular values of the parameters (γ, e, and the utility functions) such that the exogenous population profile given by ρ induces exactly zero aggregate savings in the Golden Rule steady state.

Agents with different patience levels

In the following subsection, we present an extended example with constant population growth which can be fully, analytically solved. In the next section, we will return to this example but with endogenous population proportions, with the objective of comparing the results with and without endogenous heterogeneity.

We postulate an economy in which agents vary in their level of patience. Concretely, agents are of finitely many types, $h \in H$, displaying (pairwise different) levels of patience $0 \leq \beta^h \leq 1$. Agents have an intertemporal utility function

$$u^i(c_0, c_1) = u(c_0) + \beta^h u(c_1),\ h \in H$$

where c_0, c_1 are the consumption levels in the first and second period of their lives. We take the instantaneous utility to be $u(c) = \ln(c)$, which means the intertemporal utility

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[3] Equal-treatment optimality is trivial to prove (see Azariadis (1993)). For more general results on optimality, see e.g. Malinvaud (1953) and Balasko and Shell (1980).

[4] With homogeneous preferences, the no-trade steady states are suboptimal in the Samuelson case and optimal in the Classical case—see Gale (1973).

is of the Cobb-Douglas type.

There is an exogenous gross population growth rate $\gamma > 0$. The proportion of agents with patience level β^h at time t is ρ_t^h and the population profile at time t is $\boldsymbol{\rho}_t = \left(\rho_t^h\right)_{h\in H}$.

For the given utility function, the first order conditions are necessary and sufficient for an optimum of the consumer's problem (1). We obtain:

$$c_1^h(t+1) = \beta^h R_t c_0^h(t).$$

Substituting into the budget constraint, we obtain[5]

$$c_0^h(t) = \frac{R_t + e}{R_t\left(1+\beta^h\right)}, \quad c_1^h(t+1) = \frac{\beta^h\left(R_t + e\right)}{\left(1+\beta^h\right)} \tag{5}$$

Dividing (2) by N_{t-1},

$$\gamma + e = \gamma \cdot \sum_{h\in H}\left(\rho_t^h \frac{R_t + e}{R_t\left(1+\beta^h\right)}\right) + \sum_{h\in H}\left(\rho_{t-1}^h \frac{\beta^h\left(R_{t-1}+e\right)}{\left(1+\beta^h\right)}\right)$$

and solving for R_t, using the notation $M(\boldsymbol{\rho}) = \sum_{h\in H} \rho^h \frac{1}{\left(1+\beta^h\right)}$,

$$\begin{aligned}
\gamma + e &= \gamma\left(\frac{R_t+e}{R_t}\right) M(\boldsymbol{\rho}_t) + \left(R_{t-1}+e\right)\left(1 - M\left(\boldsymbol{\rho}_{t-1}\right)\right) \Leftrightarrow \\
\gamma\left(R_t + e\right) M(\boldsymbol{\rho}_t) &= R_t\left[\gamma + e - \left(R_{t-1}+e\right)\left(1 - M\left(\boldsymbol{\rho}_{t-1}\right)\right)\right] \Leftrightarrow \\
R_t & \quad \frac{\gamma e M(\boldsymbol{\rho}_t)}{\gamma(1 - M(\boldsymbol{\rho}_t)) + e - \left(R_{t-1}+e\right)\left(1 - M\left(\boldsymbol{\rho}_{t-1}\right)\right)}
\end{aligned}$$

Solving for R_t, we obtain the current interest rate as a function of the prior interest rate R_{t-1}, and the population profile in the economy at times t and $t-1$:

$$R_t = \frac{\gamma e M(\boldsymbol{\rho}_t)}{\gamma(1 - M(\boldsymbol{\rho}_t)) + e - \left(R_{t-1}+e\right)\left(1 - M\left(\boldsymbol{\rho}_{t-1}\right)\right)} = F\left(R_{t-1}, \boldsymbol{\rho}_t, \boldsymbol{\rho}_{t-1}\right) \tag{6}$$

A sequence of interest rates $\mathbf{R} = \left(R_t\right)_{t\in\mathbb{Z}}$ verifying (6), and the corresponding consumption plans given by (5) defines an equilibrium.

If, taking a computational approach, we specify an initial condition $R_0 > 0$ and use (6) to generate a sequence of interest rates, we must observe that it is theoretically possible that $F\left(R_{t-1}, \boldsymbol{\rho}_t, \boldsymbol{\rho}_{t-1}\right) < 0$ for some t. This would imply that there exists no equilibrium corresponding to the specified initial condition.[6] It is important to take this difficulty into account because our aim is to use (6) (and analogous equations) as a reduced form for the (equilibrium) dynamics. In short, we can discard paths such that $F\left(R_{t-1}, \boldsymbol{\rho}_t, \boldsymbol{\rho}_{t-1}\right)$ becomes negative.

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[5] Given $\beta^h < \beta^{h'}$, it follows that $c_0^h(t) > c_0^{h'}(t)$ and $c_1^h(t+1) < c_1^{h'}(t+1)$ for all t, i.e. more patient agents always save more.

[6] Note that, for very particular utility functions, it might be possible that the consumption plans given by (5) coincide with the inital endowments and $F\left(R_{t-1}, \lambda_t, \lambda_{t-1}\right) = 0$ is indeed part of an equilibrium path (if we allow for zero gross interest rates in equilibrum). In this case, as noted before, the path is necessarily in the Garden of Eden case.

The constant proportions case: steady states

Consider now the constant proportions case, $\rho_t = \rho$ for all t. By Corollary 2, there are two types of steady states. In the Golden Rule steady state, we have $R^{GR} = \gamma$. In the no-trade steady state, we have

$$A_t = 1 - \left(\frac{R_t + e}{R_t}\right) M(\rho) = 0,$$

implying that

$$R^{NT}(\rho) = e\frac{M(\rho)}{1 - M(\rho)}. \tag{7}$$

In the space (R, ρ), this yields two manifolds of steady states (omitting the corresponding consumption plans), the "Golden Rule" manifold given by (γ, ρ), and the one given by $\left(\frac{eM(\rho)}{1-M(\rho)}, \rho\right)$.

Consider the monomorphic states $\rho = \delta_h$, i.e. the profiles corresponding to homogeneous populations ($\delta_h^h = 1$, $\delta_h^{h'} = 0$ for all $h' \neq h$). In this case, we simply recover the result for an OLG model with a homogeneous population, that is, in addition to the Golden Rule steady state there exists another steady state which corresponds to the no-trade equilibrium, given by

$$R^h = R^{NT}(\delta_h) = \frac{eM(\delta_h)}{1 - M(\delta_h)} = \frac{e\left(\frac{1}{1+\beta^h}\right)}{\frac{\beta^h}{1+\beta^h}} = \frac{e}{\beta^h}.$$

We refer to R^h as the *autarkic interest rate.* This is the interest rate yielding no intergenerational trade in a population where all agents are of type h. As an aside, we can give an interpretation of equation (7) as follows. Note that $1 + \beta^h = 1 + (e/R^h)$ is the present value (for a young agent) at the autarkic interest rate of the full endowment. Hence, $\frac{1}{1+\beta^h}$ is *period 1's wealth share*, that is, the share of an agent's total life wealth which is received as youth's endowment, valued at the autarkic interest rate. The quantity $M(\rho)$, which has been quite useful to simplify the analysis, turns out to be the average period 1's wealth share in the population (under a virtual segregation in which each type values its endowments at its own autarkic interest rate). Analogously, $1 - M(\rho)$ is the average period 2's wealth share. In the no-trade steady state,

$$\frac{e}{R^{NT}(\rho)} = \frac{1 - M(\rho)}{M(\rho)}.$$

i.e. the present value (in period 1) of period 2's endowment is equal to the value of period 2's average wealth share relative to period 1's average wealth share.

Observe that the intersection of the two manifolds of steady states is the set of profiles ρ^* such that

$$\frac{eM(\rho^*)}{1 - M(\rho^*)} = \gamma$$

i.e.

$$M(\rho^*) = \frac{\gamma}{e + \gamma}. \tag{8}$$

It can be easily seen that this intersection is nonempty if and only if

$$\gamma\beta^{\min} < e < \gamma\beta^{\max}$$

where $\beta^{\min} = \min_{h\in H} \beta^h$ and $\beta^{\max} = \max_{h\in H} \beta^h$ are the minimum and maximum patience level in the population.[7] Alternatively, this condition becomes

$$R^{\max} < \gamma < R^{\min}$$

where $R^{\max}$ and $R^{\min}$ are the autarkic interest rates for $\beta^{\max}$ and $\beta^{\min}$, respectively. This condition means that the endowments are such that more patient agents need to save and less patient agents are able to borrow and return the loan afterwards (there are both creditors and debtors in the economy).

We call these distinguished steady states the *Edenist Steady States*, and we call the corresponding $\boldsymbol{\rho}^*$ *Edenist profiles*.[8] These are the only Golden Rule steady states involving no intergenerational trade, i.e. $A_t = 0$. Alternatively, these are the only no-trade steady states which are also Pareto optimal. Notice that, if the proportion of patient agents is given exactly by a $\boldsymbol{\rho}^*$ satisfying (8), the argument behind Samuelson's Impossibility Theorem would fail, and convergence to the no-trade steady state would be the same as convergence to the optimal, Golden Rule state.

Consider the case with only two types of agents, $H = \{P, I\}$ with $\beta^I < \beta^P$. We call the agents with patience level β^P patient agents, and the others impatient. In this case, if there is an Edenist steady state, it must be unique. The condition for its existence is given by

$$\gamma\beta^I < e < \gamma\beta^P.$$

This is illustrated in Figure 1.

The constant proportions case: dynamics

We proceed now to analyze the dynamics of the model with exogenous, constant proportions. For general dynamic results in Overlapping Generations models, see e.g. Kehoe and Levine (1984,1990) .With constant $\boldsymbol{\rho}_t = \boldsymbol{\rho}$, the dynamics is given by

$$R_{t+1} = \tilde{F}(R_t) \tag{9}$$

where $\tilde{F}(R) = F(R, \boldsymbol{\rho}, \boldsymbol{\rho})$. It is easily shown that

$$\tilde{F}(R) = \frac{\gamma R^{NT}(\boldsymbol{\rho})}{\gamma + R^{NT}(\boldsymbol{\rho}) - R}$$

whenever the denominator is not zero, where $R^{NT}(\boldsymbol{\rho}) = \frac{eM(\rho)}{1-M(\rho)}$ as above. The steady states are given by

$$R = \gamma \text{ and } R = R^{NT}(\boldsymbol{\rho}).$$

[7] Notice that $M(\boldsymbol{\rho})$ is a linear function of ρ_h, increasing in all arguments, and therefore its maximum is reached at $\boldsymbol{\rho} = \boldsymbol{\delta}_{\min}$ and its minimum at $\boldsymbol{\rho} = \boldsymbol{\delta}_{\max}$.

[8] The set of Edenist profiles is a polyhedron given by the intersection of the simplex with the hyperplane given by (8).

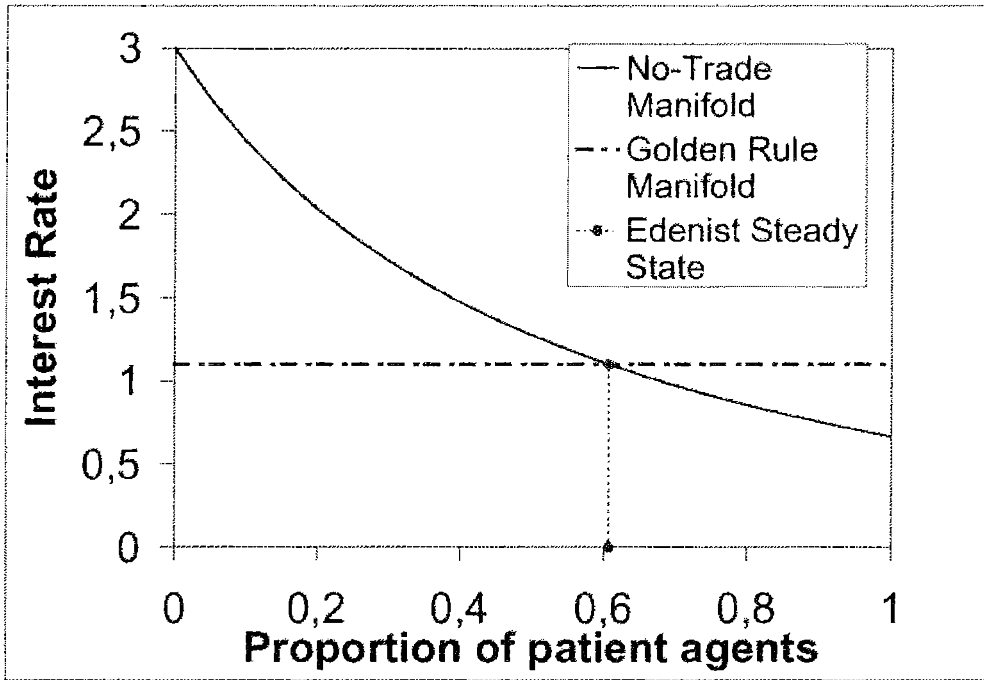

Figure 1: $\gamma = 1.1$, $e = 0.6$, $\beta^P = 0.9$, $\beta^I = 0.2$

Computing the derivative of $\tilde{F}(R)$ we obtain

$$\tilde{F}'(R) = \frac{\gamma R^{NT}(\rho)}{[\gamma + R^{NT}(\rho) - R]^2}$$

Therefore, $\tilde{F}'(\gamma) = \frac{\gamma}{R^{NT}(\rho)}$ and $\tilde{F}'(R^{NT}(\rho)) = \frac{R^{NT}(\rho)}{\gamma}$. Recall that $R^{NT}(\rho^*) = \gamma$ if ρ^* is an Edenist profile. Notice that $R^{NT}(\rho)$ can be seen as an increasing function of $M(\rho)$, hence we have $R^{NT}(\rho) > \gamma$ if and only if $M(\rho) > M(\rho^*) = \frac{\gamma}{e+\gamma}$.

Taking this observation into account, the stability analysis of the dynamical system (9) is as follows.

- If $M(\rho) > \frac{\gamma}{e+\gamma}$, then the Golden Rule steady state $R = \gamma$ is locally stable ($\tilde{F}'(\gamma) < 1$) and the no-trade steady state $R = R^{NT}(\rho)$ is unstable[9] ($\tilde{F}'(R^{NT}(\rho)) > 1$).
- If $M(\rho) < \frac{\gamma}{e+\gamma}$, then the steady state $R = \gamma$ is unstable ($\tilde{F}'(\gamma) > 1$) and the other $R = R^{NT}(\rho)$ is locally stable ($\tilde{F}'(R^{NT}(\rho)) < 1$).
- If $\rho = \rho^*$ is an Edenist profile, there is a unique steady steady, which has the following properties ($\tilde{F}'(\gamma) = \tilde{F}'(R^{NT}(\rho^*)) = 1$). If $R_0 \leq \gamma$, the system converges to γ. If $R_0 > \gamma$ the system diverges from γ. In the latter case, the economy eventually collapses and we obtain $\tilde{F}(R_t) < 0$, meaning that the path does not constitute an

[9] Paths starting at interest rates slightly lower than $R^{NT}(\rho)$ converge to $R = \gamma$, while paths starting at interest rates slightly higher than $R^{NT}(\rho)$ diverge, meaning that they cannot constitute equilibrium paths.

equilibrium.This is illustrated in Figure 2.

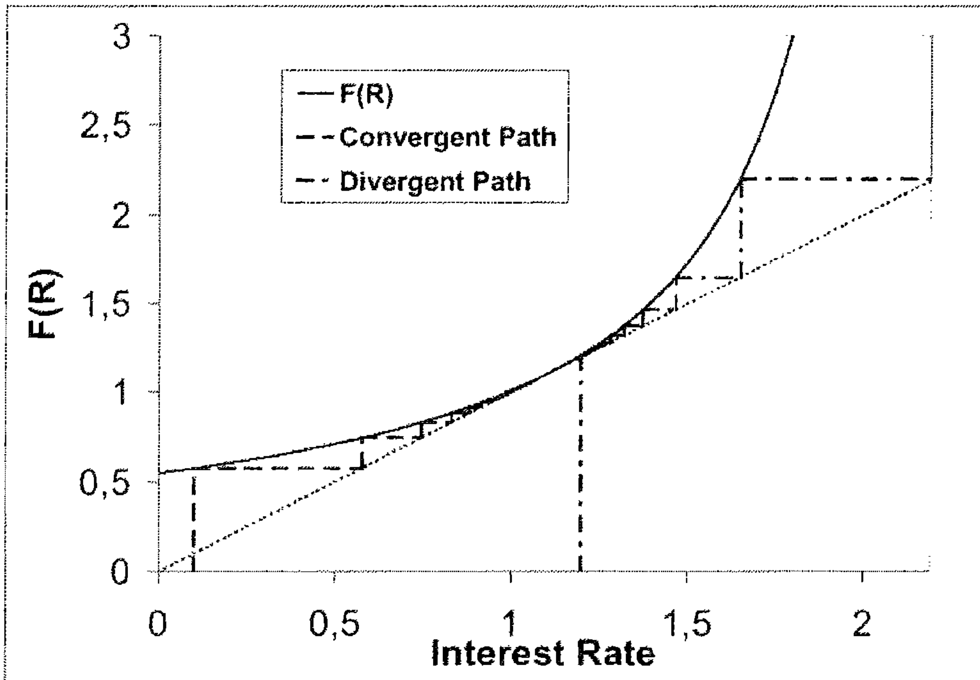

Figure 2: The dynamics for an Edenist profile with $\gamma = 1.1$.

We can now compare our results with the homogenous agents case studied by Gale (1973). The aggregate savings at the Golden Rule steady state are

$$A(\rho) = \sum_{h \in H} \rho^h \left(1 - \frac{\gamma + e}{\gamma\left(1 + \beta^h\right)}\right) = 1 - \frac{\gamma + e}{\gamma} M(\rho)$$

Gale (1973) classifies the OLG economies according to the aggregate savings at the corresponding Golden Rule steady states. That is, an economy is in a *Samuelson case* if $A(\rho) > 0$, and is in a *Classical case* if $A(\rho) < 0$. $A(\rho) = 0$ if and only if $\rho = \rho^*$ is an Edenist profile.Clearly, $A(\rho) < (>) 0$ if and only if $M(\rho) > (<) \frac{\gamma}{e+\gamma}$. Therefore, the stability conditions of the model with heterogeneous agents are analogous to the conditions of the model with homogeneous agents. That is, the Golden Rule is locally stable if $A(\rho) < 0$ (Classical case), and unstable if $A(\rho) > 0$ (Samuelson case).

As an illustration, consider again the case with only two types of agents, $H = \{P, I\}$ with $\beta^I < \beta^P$. Assuming that $\gamma\beta^I < e < \gamma\beta^P$, there is a unique Edenist profile ρ^*. Notice that $M(\rho) = M(\rho^P, \rho^I) = M(\rho^P, 1 - \rho^P)$ is a decreasing function of ρ^P, and hence the classification sketched above is summarized as follows. If ρ^P is smaller than the proportion of patient agents in ρ^*, the economy is in the Classical case, with a stable Golden Rule, while if ρ^P is larger, the economy is in the Samuelson case with an unstable Golden Rule.

OLG with endogenous heterogeneity

Our aim in this section is to present a model of *selection of preferences* by endogenizing the proportion of agents endowed with different preferences in the previous model, that is, by endogenizing heterogeneity.

Models for the evolution or selection of preferences (see L. Samuelson (2001) for an overview) are generally based on the darwinian idea that preferences leading to a higher *fitness* (success) spread in the population, at the expense of others. Quite naturally, this approach raises two questions. The first one is the difference between this fitness and the preferences or payoffs themselves. The second one is how exactly is this darwinian idea modeled, that is, which kind of dynamics is assumed. We now tackle these two questions in our framework.

The first question is a tough one in, e.g., a game-theoretic setting. In the original, biological models, fitness is completely identified with payoffs; e.g., the payoffs in a given game can be taken to represent the expected number of offspring. Game theory, in turn, identifies payoffs with agent's preferences. Consequently, it is not possible to use fitness (payoffs) as a criterion for the selection of preferences (payoffs). The *indirect evolutionary approach* (see Güth and Yaari (1992) and Güth (1995)) solves this problem by endowing players with preferences over the outcomes of the game which do not need to coincide with the underlying fitnesses associated to those outcomes. Quoting L. Samuelson (2001), "agents whose preferences lead to high fitnesses tend to reproduce faster than those with lower fitnesses, either in a literal biological sense or in a figurative learning sense".

In our setup, preferences are explicitly given through the utility functions u^h. It is easy to argue, though, that these preferences are in principle not observable. The success of other agents is rather inferred from observable magnitudes like consumption levels. Accordingly, we will measure the (observed) success of a given type of agents not by its attained utility levels, but by its (average) consumption. We will then interpret that more successful preferences spread through acculturation of the new generation (not through any differences in fertility).

The second question (the precise form of the dynamics) has given rise to a large literature in evolutionary game theory (for an introduction to the topic, see Friedman (1991, 1998) or the books by Weibull (1995) or Hofbauer and Sigmund (1998)). The most basic evolutionary dynamics is given by the Replicator Equation, which postulates that the growth rates of the different types of agents are exactly proportional (or equal, choosing an appropriate time scale) to their average payoffs (taken as fitness).[10] Consequently, types whose payoffs are higher than the average population payoff spread at the expense of other types. For the continuous time case, this dynamics has been generalized (see Ritzberger and Weibull (1995)) to the class of *monotonic dynamics*, which are dynamics such that the growth rate of any given type is larger than that of another type if and only if the average payoff of the former is larger than that of the latter. A further generalization leads to the concept of *Sign-Preserving Selection* (SPS) dynamics, which are defined by the property that types obtaining higher payoffs than

[10] With an economic interpretation in mind, [?] calls such dynamics "Malthusian".

the average are the ones growing at nonnegative rates (see again Ritzberger and Weibull (1995)).

In an Overlapping Generations model, time is discrete. We consider a class of discrete-time dynamics analogous to the game-theoretic SPS dynamics. This class will be defined by the property that the proportion of agents of a given type grows if and only if that type's average consumption level is higher than the population's average consumption level.

The dynamics with endogenous proportions

Consider the case of constant population growth rate, $N_t = \gamma N_{t-1}$ for all t. Denote by $CM^h(t)$ the average consumption among the individuals of type $h \in H$ living at time t (that is, including both young and old individuals).

$$\begin{aligned} CM^h(t) &= \frac{1}{(N_t^h + N_{t-1}^h)}\left[N_t^h c_0^h(t) + N_{t-1}^h c_1^h(t)\right] \quad (10)\\ &= \frac{1}{(\gamma\rho_t^h + \rho_{t-1}^h)}\left[\gamma\rho_t^h c_0^h(t) + \rho_{t-1}^h c_1^h(t)\right] \end{aligned}$$

which is well defined whenever ρ_t^h or ρ_{t-1}^h are different from zero. Denote by $CM(t)$ the average consumption among all individuals living at time t (again including both young and old individuals).

$$\begin{aligned} CM(t) &= \frac{1}{(N_t + N_{t-1})}\left[\sum_{h\in H} N_t^h c_0^h(t) + \sum_{h\in H} N_{t-1}^h c_1^h(t)\right] = \\ &\frac{1}{(\gamma+1)}\left[\sum_{h\in H} \gamma\cdot\rho_t^h c_0^h(t) + \sum_{h\in H} \rho_{t-1}^h c_1^h(t)\right] \end{aligned}$$

If the market clearing condition (2) holds, it follows that the average consumption in the population for $t \geq 1$ is constant:

$$CM(t) = \frac{e+\gamma}{1+\gamma} \quad (11)$$

We consider the class of dynamics such that $\rho_{t+1}^h > \rho_t^h$ (respectively $<$ or $=$) if and only if $CM^h(t) > CM(t)$ (respectively $<$ or $=$). As commented before, this corresponds to the analogue of Sign-Preserving Selection dynamics in our setup.

Let $\mathcal{E}^* = \left(e, \{u^h\}_{h\in H}, (N_t)_{t\in\mathbb{Z}}\right)$ denote an economy with endogenous heterogeneity. Although the following definition is stated for the general case, we will concentrate on economies with constant population growth rate.

Definition 3 *An* equilibrium *for the economy $\mathcal{E}^*$ is a triple $\left(\mathbf{C}, \mathbf{R}, (\boldsymbol{\rho}_t)_{t\in\mathbb{Z}}\right)$ where*

(i) $(\boldsymbol{\rho}_t)_{t\in\mathbb{Z}} = \left\{(\rho_t^h)_{h\in H}\right\}_{t\in\mathbb{Z}}$ is a sequence of population profiles,

(ii) $(\mathbf{C}, \mathbf{R})$ is an equilibrium for the economy $\left(e, \{u^h\}_{h\in H}, (N_t, \boldsymbol{\rho}_t)_{t\in\mathbb{Z}}\right)$ in the sense of

Definition 1, and

(iii) for all $t \in \mathbb{Z}$, the following condition is satisfied:

$$sign\left(\rho_{t+1}^h - \rho_t^h\right) = sign\left(CM^h(t) - CM(t)\right)$$

and the boundary condition $\rho_{t+1}^h = 0$ whenever $\rho_t^h = \rho_{t-1}^h = 0$.

An equilibrium is an envy-free steady state *if* $\boldsymbol{\rho}_t = \boldsymbol{\rho}$ *for all* t, *and* $\left(c_0^h(t), c_1^h(t+1)\right) = \left(c_0^h, c_1^h\right)$ *for all* t *and all* $h \in H$.

The name *envy-free* arises from the fact that, in such a steady state, condition (iii) implies that average consumption is constant across types. Although an envy-free steady state for the economy $\mathcal{E}^*$ is necessarily a steady state for the associated economy with exogenous (and constant) proportions, in the sense of Definition 1, the converse needs not be true.

In general, an equilibrium would define a dynamic path as follows. As in equation (6), through the market-clearing condition and agents' demands we obtain R_t as a (possibly implicit) function of R_{t-1}, $\boldsymbol{\rho}_t$, and $\boldsymbol{\rho}_{t-1}$. Further, a sign-preserving dynamics will specify $\boldsymbol{\rho}_{t+1}$ as a function of $CM^h(t)$ and $CM(t)$. The latter depend only on R_t, R_{t-1}, $\boldsymbol{\rho}_t$, and $\boldsymbol{\rho}_{t-1}$. Since R_t is a function of the other three quantities, we obtain a (possibily implicit) **reduced form of the dynamics**:

$$\begin{aligned} R_t &= F\left(R_{t-1}, \boldsymbol{\rho}_t, \boldsymbol{\rho}_{t-1}\right) \\ \rho_{t+1} &= G\left(R_{t-1}, \boldsymbol{\rho}_t, \boldsymbol{\rho}_{t-1}\right) \end{aligned} \tag{12}$$

This reduced form reflects a natural way of interpreting the model. Both R_t and ρ_{t+1} are determined in period t. The interest rate R_t is determined through the market clearing condition and the demands of the agents living at t. The proportion of patient agents in generation $t+1$ is determined at the end of period t by the relative success of the agents living at t (young and old). Our interpretation is that individuals of generation $t+1$ are already alive (as children) at t and accultured in this period. Individual agents tend to acquire the behavior (or preferences) of more successful agents with higher probability adn, thus, in the aggregate, more successful preferences spread at the expense of others. We remark again that this does not imply any differences in fertility.

Steady States

Consider an economy with constant population growth rate γ and an envy-free steady state given by a collection of consumption plans (constant in time) $\mathbf{C}$ and a constant population profile $\boldsymbol{\rho}$. The collection of consumption plans $\mathbf{C}$ is also a steady state (in the sense of Definition 1) of the associated economy with constant but exogenous proportions given by $\boldsymbol{\rho}$. By Corollary 2, we know that either the interest rate in the envy-free steady state is constant and equal to γ, or the aggregate savings A_t are always identically zero. The next theorem shows that, actually, the second case can only happen in degenerate situations, and hence the only relevant envy-free steady states are the Golden Rule ones.

Theorem 3 *Let* $\mathcal{E}^* \left(e, \{u^h\}_{h\in H}, (N_t)_{t\in\mathbb{Z}}\right)$ *be an economy with constant population growth rate* γ. *An equilibrium* $(\mathbf{C}, \mathbf{R}, \boldsymbol{\rho})$ *is an envy-free steady state if and only if either*

(i) $R_t = \gamma$ and $\left(c_0^h(t), c_1^h(t+1)\right) = \left(c_0^h, c_1^h\right)$ for all t, or
(ii) $A^h = 1 - c_0^h = 0$ for all types h such that $\rho_h > 0$.

Proof. By Definition3(iii), in an envy-free steady state $CM^h = CM$ for all h where CM^h is well-defined, i.e. $\rho_h > 0$. Therefore, using (10) and (11),

$$\frac{1}{\gamma\rho_h + \rho_h}\left(\gamma\rho_h c_0^h + \rho_h c_1^h\right) = \frac{e+\gamma}{1+\gamma}$$

or, equivalently,

$$\gamma A^h = \gamma(1 - c_0^h) = c_1^h - e. \tag{13}$$

From the budget constraint for agents of type h, we know that

$$R_t A^h = c_1^h - e.$$

Suppose there is some type h with $\rho_h > 0$ and $A^h \neq 0$. Then, it follows that $R_t = \gamma$ for all t and the envy-free steady state is of the Golden Rule type. The only remaining case is that $A^h = 0$ for all types h with $\rho_h > 0$. The converse implication is obvious.[11] ■

We want to argue that the relevant envy-free steady states are those corresponding to the Golden Rule $R_t = \gamma$, that is, type (i) in the last Theorem. To see this, consider an envy-free steady state of type (ii). If there is only one type with $\rho_h > 0$, then we must have that $\boldsymbol{\rho} = \boldsymbol{\delta}_h$, the monomorphic state where all agents are of type h and hence there is no heterogeneity. Suppose there is more than one type of agents with $\rho_h > 0$. Since $A^h = 0$ for all such types, it follows that all of them maximize their utility functions at the same interest rates at the consumption plan $(1, e)$. For non-zero gross interest rates, this can only happen for pathological, non-generic cases where different utility functions are tangent to the same budget constraint at exactly the same consumption bundle, namely $(1, e)$. For instance, consider the example with difference patience levels as in Subsection 2.3. The Marginal Rate of Substitution for type h is given by $(1/\beta^h)u'(c_0)/u'(c_1)$, and hence different types have different MRS at all consumption plans. It follows that the pathological cases described above can never happen in this example.

We interpret Theorem 3 as follows. In the model without endogenous heterogeneity, there are two types of steady states, the Golden Rule ones and the no-intergenerational-trade steady states. Once we endogenize heterogeneity, the latter steady states disappear. In other words, once we add an additional dimension, in those states the population distribution of types does not remain steady, and the system moves away. In the interior of the state space, (save degenerate cases) only the Golden Rule states remain.

This result has several important implications.

[11] The requirement of constant consumption plans is obviously only necessary for the converse implication. It can be dropped from the statemente if the utility functions are strictly quasiconcave. Otherwise, the consumers' optimization problems might have multiple solutions at interest rate γ, which might yield equilibria with constant interest rate but varying consumptions.

Consequence 1: Optimality. Suppose that the dynamics on the proportion of agents is such that convergence to an interior population profile obtains. Then, except maybe in degenerate cases as commented above, the system must converge to an envy-free steady state of the Golden Rule type, and hence Pareto optimal. This is in stark contrast to the basic Overlapping Generations model, where there might be convergence to the suboptimal no-trade steady states (this obtains e.g. in the Samuelson case).

Consequence 2: Garden of Eden. A second implication is that Samuelson's Impossibility Theorem does not necessarily hold any more. Recall that, if the economy is in the Garden of Eden case, necessarily $A_t = 0$ for all t (by Corollary 2). With constant, exogenous proportions, in general the Golden Rule steady state will entail $A_t \neq 0$ (and constant). Hence, the Garden of Eden economy cannot converge to the optimal, Golden Rule steady state. With endogenous heterogeneity, though, if the Garden of Eden economy converges to an interior point, then it must do so to an Edenist steady state where $A_t = 0$ and the interest rate is given by the population growth rate. That is, the population proportions will adjust in such a way that the system reaches an optimal, Golden Rule steady state with $A_t = 0$.

Some comments are in order. The fact that equilibria in OLG models are, in general, suboptimal, has been regarded as a puzzle. One explanation is the "lack of market clearing at infinity" (see e.g. Geanakoplos (1989)). We think that Theorem 3 and Consequence 1 might shed light on this problem. Observe that, adding a new, natural dimension to the model, optimality obtains generically, in the sense explained above: provided the system converges, it must do so to a Godlen Rule state, for almost all parameters of the model. Consequence 2 can then be seen as another illustration where endogenizing heterogeneity in the OLG model allows us to better understand a paradoxical result of the original model.

Remark 1 *Notice that, in any envy-free steady state, equation (13) implies that*

$$\gamma\left(1 - c_0^h(t)\right) = c_1^h(t) - e$$

and hence

$$N_t\left(1 - c_0^h(t)\right) = N_{t-1}\left(c_1^h(t) - e\right)$$

for all types actually present in the population. These equations could be interpreted as if *agents of type h traded only among themselves. Even under such an interpretation, though, the market prices are determined by the fact that there are several types in the population.*

Endogenous heterogeneity with different patience levels

We return now to the example with different patience levels as in Subsection 2.3. Instead of assuming exogenous population proportions, we allow now ρ_t to evolve according to a Sign-Preserving Selection dynamics in the sense of Definition 3(iii).

Steady States

Without dynamics in the population distribution of types, we know that there are two possible manifolds of steady states: Golden Rule equilibria with $R = \gamma$ and no-trade equilibria with $R^{NT}(\boldsymbol{\rho})$ given by (7). Since any envy-free steady state in the sense of Definition 3 induces a steady state for the associated population profile in the sense of Definition 1, it follows immediately that the only candidate envy-free steady states are also the Golden Rule and the no-trade ones. The next result shows that, surprisingly, only the former are indeed steady states.

Corollary 4 *The envy-free steady states of the model with different patience levels are the Golden Rule states with $R = \gamma$ (for any population profile $\boldsymbol{\rho}$), i.e. any proportion of patient agents provided that the interest rate is equal to the population growth rate (the biological rate of interest), plus the monomorphic steady states where $\boldsymbol{\rho} = \boldsymbol{\delta}_h$, with interest rate $R^h = \frac{e}{\beta^h}$ (the no-trade degenerate steady states).*

Proof. We know from Subsection 2.3 that, in an envy-free steady state, either

$$R = \gamma \text{ or } R = R^{NT}(\boldsymbol{\rho}) = \frac{eM(\boldsymbol{\rho})}{1 - M(\boldsymbol{\rho})}.$$

From Theorem 3(ii), though, we obtain that, in any envy-free steady state with $R \neq \gamma$, for any type h actually present in the population,

$$A^h = 1 - c_0^h = 0$$

which, using 5, translates into

$$\frac{R_t + e}{R_t\left(1 + \beta^h\right)} = 1$$

i.e.

$$R_t = \frac{e}{\beta^h} = R^h.$$

Moreover, since, necessarily

$$R^h = R^{NT}(\boldsymbol{\rho}) = \frac{eM(\boldsymbol{\rho})}{1 - M(\boldsymbol{\rho})}$$

it follows that

$$M(\boldsymbol{\rho}) = \frac{1}{1 + \beta^h}$$

implying (since $M(\boldsymbol{\rho})$ is the convex combination of the quantities $\frac{1}{1+\beta^k}$), that $\rho_h = 1$ and $\boldsymbol{\rho} = \boldsymbol{\delta}_h$. ■

In summary, in the space $(R, \boldsymbol{\rho})$ we have two manifolds of "static" steady states, the "Golden Rule" manifold given by $(\gamma, \boldsymbol{\rho})$, and the no-trade one given by $\left(\frac{eM(\boldsymbol{\rho})}{1-M(\boldsymbol{\rho})}, \boldsymbol{\rho}\right)$. If we now add any Sign-Preserving Selection dynamics on $\boldsymbol{\rho}$, allowing the proportions of agents with different patience levels to change over time, what Theorem 4 says is that the second manifold disappears. In other words, once we add an additional dimension, in those states the population proportions of types do not remain steady, and the system moves away. In the interior of the state space $(R, \boldsymbol{\rho})$, only the manifold of optimal Golden Rule states remains. In all these steady states, the interest rate is equal to the growth rate of the population. In the boundaries, we simply recover the result for an OLG model with a homogeneous population, that is, in addition to the Golden Rule states there are isolated, steady states which correspond to the degenerate "no-trade equilibria." One way to interpret Corollary 4 is that the no-trade equilibria disappear if the population is heterogeneous, no matter how small the heterogeneity, and intertemporal efficiency (via the Golden Rule) is guaranteed (recall Consequence 1). The only no-trade, non-degenerate steady states which remain are those which are also Golden Rule states, i.e. the Edenist states $\boldsymbol{\rho}^*$.

Consider again the case with only two patience levels, $\beta^I < \beta^P$, illustrated in Figure 1. With dynamics in $\boldsymbol{\rho}$, the envy-free steady states are those corresponding to the Golden Rule manifold and the two extremes of the no-trade manifold. The latter is now not a manifold of steady states, but rather contains (by Corollary 2) all Garden of Eden paths. Provided that $\gamma\beta^I < e < \gamma\beta^P$, there exists a unique envy-free steady state which is simuyltaneously of the no-trade and the Golden Rule type, namely the Edenist state, which can be explicitly computed (using (8)) as

$$(\rho^*)^P = \left(\frac{1+\beta^P}{\beta^P - \beta^I}\right)\left(\frac{e-\gamma\beta^I}{e+\gamma}\right). \tag{14}$$

Dynamics

Stability Analysis: comments

In the general case, the OLG economy with endogenous heterogeneity evolves according to a discrete, multi-dimensional dynamical system (see e.g. Kelley (1991) for an introduction). We know that there is a connected continuum of steady states (the Golden Rule states), which implies that none of them can be, strictly speaking, asymptotically stable. Yet, some of them might be Lyapunov stable, and it might even be the case that almost all trajectories converge to a single steady state. Linearization, though, will in general be fruitless, since whenever there is a (smooth), at least one-dimensional manifold of steady states, the Jacobian matrix of the dynamics at those states has a unit eigenvalue, violating the hyperbolicity hypothesis which allow for an analysis through linearization.[12] The presence of a unit eigenvalue is very bad news for a linearization attempt. If there are other eigenvalues of modulus strictly greater than unity, we would conclude from linearization that the state is unstable. If all other eigenvalues have moduli smaller than or equal to one, the state is not hyperbolic (in the sense of discrete dynamical systems) and linearization fails, that is, no conclusion can

[12] To see this, let the dynamics be given by a vectorial function Φ and let the manifold of steady states be parameterized as $s(\lambda)$. We have that $\Phi(s(\lambda)) = s(\lambda)$, implying that $J\Phi(s(\lambda)) \cdot \nabla s(\lambda) = \nabla s(\lambda)$, where $J\Phi$ denotes the Jacobian matrix and ∇s denotes the gradient. Therefore, $J\Phi$ has a unit eigenvalue. If the parameter λ where k-dimensional, this eigenvalue would have multiplicity k.

be derived about local stability (see e.g. Kelley (1991)).

Since linearization might fail in part of the Golden Rule manifold, the next natural step would be to attempt to find Lyapunov functions to study the global behavior of the system. We have not been able to find any such Lyapunov function. Since we have not specified a concrete dynamics, no other meaningful stability analysis can be performed at this point. Below we will present an example of such a dynamics, whihc will in turn allow us to perform simulations for particular instances of the model.

An example of an SPS dynamics

We proceed, thus, to give an example of an SPS dynamics, with the aim of performing simulations for a specific instance of the model.

In the framework of Evolutionary Game Theory, the benchmark dynamics is the Replicator Dynamics (see Hofbauer and Sigmund (1998) or Starret (1972)). The various types of (well-behaved) dynamics present in the literature can be understood as generalizations of the Replicator Dynamics. According to the latter, the growth rate of any strategy should be (in continuous time) numerically equal to the difference between the average payoff of agents employing the considered strategy and the average payoff in the whole population, divided then by the population average payoff (the latter is a mere normalization). A direct, naive translation of this dynamics into the current framework would yield

$$\rho_{t+1}^h - \rho_t^h = \rho_t^h \left(\frac{CM^h(t) - CM}{CM} \right) = \rho_t^h \left(\frac{CM^h(t)}{CM} - 1 \right)$$

where $CM^h(t)$ is the average consumption in the type h individuals alive at t (recall (10)), and the population average consumption is constant in time (recall (11)), $CM(t) = CM = \frac{e+\gamma}{1+\gamma}$. This dynamics would obviously be SPS. It is, however, actually not a well-defined dynamics for our purposes, because the hyperplane

$$\sum_{h \in H} \rho_t^h = 1$$

is not forward invariant. In other words, under this dynamics, population proportions might (and do in general) cease to add up to one. The reason is simply that the weighted average of the average consumptions is not equal to the population average consumption:

$$\sum_{h \in H} \rho_t^h CM^h(t) \neq CM$$

because using the population proportions ρ_t^h we are ignoring the OLG structure of the model. The right weights are the population proportion of agents of type h, young and old, who are alive at t. Indeed,

$$\sum_{h \in H} X_t^h CM^h(t) \neq CM$$

where

$$X_t^h = \frac{1}{1+\gamma}(\gamma \rho_t^h + \rho_{t-1}^h)$$

and hence the appropriate dynamics would be given by

$$\rho_{t+1}^h - \rho_t^h = X_t^h \left(\frac{CM^h(t)}{CM} - 1 \right)$$

This form is intuitively appropriate. Agent of generation $t+1$ become accultured in period t (when they are children) and observe the relative success of type h agents, both young and old. The microeconomic foundations of standard evolutionary models involve agents sampling role models at random from the population before, say, deciding whether to adopt a different strategy from that of their parents. Hence, the increase in the proportion of agents of a given type should be proportional, not to the proportion of type h agents in generation t, but to the proportion of type h agents alive in period t.

The latter dynamics is obvioulsy SPS (notice also that $X_t^h = 0$ if and only if $\rho_{t+1}^h = \rho_t^h = 0$). Further,

$$\sum_{h \in H} \rho_{t+1}^h = \sum_{h \in H} \rho_t^h + \frac{1}{CM} \sum_{h \in H} X_t^h CM^h(t) - 1 = 1.$$

There is, however, another difficulty. As many discrete-time dynamics, this one presents overshooting problems, in the sense that the population simplex $\Delta = \left\{ \left(\rho_t^h\right) \middle| \sum_{h \in H} \rho_t^h = 1,\ 0 \leq \rho_t^h \leq \right.$ is not be forward invariant. The adjustment of the population proportion of types might occassionally be too quick and cause this proportions to overshoot 0 or 1, becoming either negative or strictly larger than one. This can be fixed adding a speed adjustment parameter k_t:

$$\rho_{t+1}^h - \rho_t^h = k_t X_t^h \left(\frac{CM^h(t)}{CM} - 1 \right). \tag{15}$$

This parameter is determined as follows: Denote $E^h(t) = X_t^h \left(\frac{CM^h(t)}{CM} - 1 \right)$. Then,

$$k_t = \min \left\{ 1, \left\{ -\frac{\rho_t^h}{E^h(t)} \mid E^h(t) < 0 \right\} \right\}.$$

This guarantees that $\rho_{t+1}^h \geq 0$ for all types, and, since $\sum_{h \in H} \rho_{t+1}^h = 1$, this implies that $\rho_{t+1}^h \leq 1$. Note, though, that $k_t < 1$ if and only if some type becomes exctinct. Alternatively, we could also pick $\hat{k}_t = \delta \cdot k_t$ as adjustment parameter, with $0 < \delta < 1$, to obtain a dynamics where extinction of types is not possible.

Simulations

We have performed simulations for the dynamics (15) in the example with two types of agents endowed with different patience levels, $\beta^I < \beta^P$, and logarithmic instantaneous utility function. These are meant to be merely illustrative, and hence we provide here only a qualitative description of our observations.

Let ρ denote here the proportion of more patient agents. The first result is that we observe almost global convergence to the Golden rule manifold, i.e. most paths converge to some state with a constant proportion of patient agents and interest rate $R = \gamma$. However, only the "lower" part of this manifold is stable. Specifically: there exists a $0 < \bar{\rho} < 1$ such that states $((\rho, \rho), R = \gamma)$ with $\rho < \bar{\rho}$ are limits of the paths of the dynamics, while the analogous states with $\rho > \bar{\rho}$ are repellors. We observe that this threshold $\bar{\rho}$ is close to but larger than the Edenist proportion ρ^* (given by (14) in this case) . Roughly speaking, this implies that Golden Rule steady states in the Classical Case ($\rho < \rho^*$) are stable, while most such states in the Samuelson Case are unstable, in agreement with the constant proportions case (Section 2).

Moreover, for most paths, convergence is well-behaved in the sense that after a few iterations (ρ_t, ρ_{t+1}) is already close to the diagonal and does not move away from it hereafter. Further, there seems to be a non-negligible set of initial conditions leading to exact convergence to the Edenist steady state ρ^*. A sample path illustrating both of these characteristics is shown in Figure 3.

For a specific subset of initial conditions with high ρ_0 but low ρ_1, we find quick "hyperinflation" phenomena with very high interest rates. The economy soon collapses and the equations, as stated, yield a negative interest rate. As observed before (Section 2.3), the interpretation is merely that for these initial conditions there is no equilibrium of the double-ended OLG economy.

For a subset of initial conditions with low ρ_0 but high ρ_1, however, we observe a relatively quick extinction of the impatient agents. In this case, the dynamics converges to the monomorphic state with only patient agents and autarkic interest rate $R = e/\beta^P$. This, however, is most probably particular of the chosen dynamics, since, as remarked above, a different choice of the adjustment parameters k_t would prevent finite-time extinction phenomena.

Conclusion

We have presented a model for the selection of preferences in an Overlapping Generations setting. In contrast to most of the literature on the evolution of preferences, which concentrates on game-theoretic models, we have hence moved to a general-equilibrium framework. By explicitly modelling consumption decisions, we are able to clearly distinguish between the object evolving (underlying preferences) and the measure of its success (observable consumption levels).

Rather than assuming a specific functional form for our dynamics, we have simply postulated a family of dynamics arising from the most general dynamics considered in evolutionary game theory. This allows us to show robustness of our findings with respect to the model specification.

The contribution of our paper is twofold. First, we add to the growing literature on the evolution of preferences in a setting not previously considered. This setting is of explicit economic nature and, further, provides a natural distinction between the object that evolves (preferences) and the fitness which drives that evolution (consumption levels). Second, we think that our results are meaningful for the literature on Overlapping Generations models. Concretely, we show that the introduction of endogenous hetero-

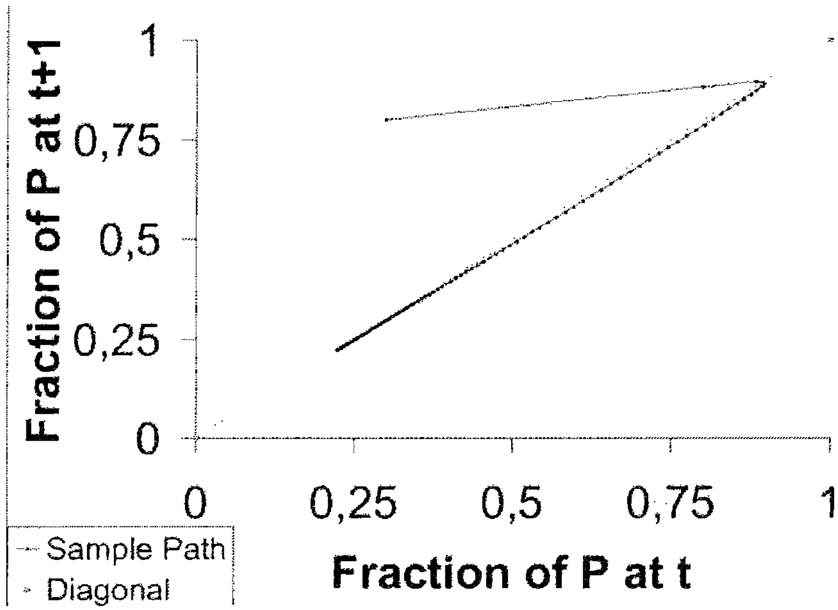

Figure 3: A typical path with $e = 0.5$, $\gamma = 1.1$, $\beta^I = 0.4$, and $\beta^P = 0.8$. The initial condition is $\rho_0 = 0.3, \rho_1 = 0.8, R_0 = 1.02$. The path approaches the diagonal almost immediately and then converges to $\rho^* = 0.16875$ and $R = \gamma$.

geneity in those models allows the selection of optimal, golden-Rule steady states and provides a solution to the paradox posed by Samuelson's Impossibility Theorem.

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