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# Stability and Equilibrium Selection in a Link Formation Game

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## *Abstract*

*In this paper we use a non cooperative equilibrium selection approach as a notion of stability in link formation games. Specifically, we first extend the global games literature, originally introduced by Carlsson and van Damme (1993), generalizing an important uniqueness result to a broader class of games: games with vector valued space of actions. Then we apply this result to study the robustness of the set of Nash equilibria for a class of link formation games in strategic form with supermodular payoff functions. Interestingly, the equilibrium selected is in conflict with those predicted by the traditional cooperative refinements. Moreover, we find a conflict between stability and efficiency even when no such conflict exists with the cooperative refinements. We discuss some practical issues that these different theoretical approaches raise in reality.*

*JEL codes: C70, D20, D82*
*Keywords: Global Games, Networks, Equilibrium Selection.*

## *Resumen*

*En este artículo introducimos un esquema no cooperativo de selección de equilibrios como una noción de estabilidad en juegos de formación de redes. Específicamente, extendemos primero un resultado de la literatura de juegos globales, originalmente introducida por Carlson y van Damme (1993), generalizando un importante resultado de unicidad a una clase más amplia de juegos: juegos con espacios de acciones multidimensionales. Luego aplicamos este resultado al estudio de cuán robusto es el conjunto de equilibrios de Nash a la introducción de información incompleta en una clase particular de juegos estáticos de formación de redes. La característica distintiva de esta clase es que las funciones de pago son supermodulares. Un resultado interesante es que se selecciona un sólo equilibrio que está en conflicto con aquellos predichos por los refinamientos cooperativos tradicionales. Además, encontramos un conflicto entre estabilidad y eficiencia aun cuando tal conflicto no existe si se usan los refinamientos cooperativos. Discutimos también algunas implicaciones prácticas asociadas a estas diferencias.*

# Stability and Equilibrium Selection in a Link Formation Game

by

Rodrigo Harrison and Roberto Muñoz *

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**Abstract**

In this paper we use a non cooperative equilibrium selection approach as a notion of stability in link formation games. Specifically, we first extend the global games literature, originally introduced by Carlsson and van Damme (1993), generalizing an important uniqueness result to a broader class of games, games with vector valued space of actions. Then we apply this result to study the robustness of the set of Nash equilibria for a class of link formation games in strategic form with supermodular payoff functions. Interestingly, the equilibrium selected is in conflict with those predicted by the traditional cooperative refinements. Moreover, we find a conflict between stability and efficiency even when no such conflict exists with the cooperative refinements. We discuss some practical issues that these different theoretical approaches raise in reality.

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## 1 Introduction

The way that different agents interact has an important role in the outcome of many problems in economics and other social sciences. Recently, these interactions have been modeled using *network structures* or *graphs*, where the agents are represented by nodes and the arcs between nodes represent some specific kind of relation between the corresponding agents. This approach has proved to be successful in the study of many specific problems;[1] however, we do not have a unique and accepted theory to explain how the networks form, which properties they have in terms of social welfare, and how robust some results are in specific environments when some of the assumptions are slightly modified. It is well known in the literature that, in general, a link formation game in strategic form can lead to the formation of multiple networks supported by multiple Nash equilibria. Even more, under some particular circumstances, any network can be supported by a Nash equilibrium of the game.[2] The use of traditional refinements is limited and depends on the details of the game, consequently, some stability notions have been used in order to refine the set of equilibria.

The stability notions used so far to refine the set of Nash equilibria in a link formation game have been based on cooperative game theory. The most prominent of them, from the strongest to the weakest, have been Strong Nash Equilibrium (SNE), Coalition Proof Nash Equilibrium (CPNE) and Pairwise Stability (PS). However, the applicability of these refinements lies critically on the feasibility of cooperation among agents. This assumptions may be too strong for a link formation game when, by definition, the network has not been formed.[3]

[1] For an excelent review of the main issues in network theory see Dutta and Jackson (2001) and Jackson (2001).

[2] See, for example, Slikker and van den Nouweland (2000).

[3] The feasibility of cooperation seems more appealing once the network has been formed and the agents interact among them.

In this paper we use a non cooperative[4] equilibrium selection approach as a notion of stability in link formation games. Specifically, we follow the *Global Games* approach pioneered by Carlsson and van Damme (1993),[5] to study the robustness of the set of Nash equilibria for a class of link formation games with supermodular payoff functions. In order to illustrate this approach, let us suppose that $G_x$ is a standard game of complete information where the payoffs depend on a parameter $x \in \mathbb{R}$, and also suppose that for some subset of the parameter $x$, $G_x$ has a strict Nash equilibrium. Rather than observing the parameter $x$, suppose instead that each player payoff function depends on a private value $x_i = x + \sigma\varepsilon_i$ where $\sigma > 0$ is a scale factor and $\varepsilon_i$ is an *i.i.d.* random variable with density $\phi$. Note that under this structure $x_i$ contains diffuse information about other players' private values. Denote this "perturbed game" by $G_x(\sigma)$, and let $NE(G_x)$ and $BNE(G_x(\sigma))$ denote the sets of Nash and Bayesian Nash equilibria of the unperturbed and perturbed games, respectively. Equilibrium selection is obtained when, conditional on $x$, the actions determined by the strategies in $\lim_{\sigma \to 0} BNE(G_x(\sigma))$ are included in $NE(G_x)$. Carlsson and van Damme (1993) show, in fact, that for two-player, two-action games, this limit comprises a single equilibrium profile, and is obtained through iterated deletion of strictly dominated strategies. Recently these results have been extended by Frankel, Morris and Pauzner (2002) for games with many players and many actions, but it is limited to the case of games with strategic complementarities and where the action space is a compact subset of the real numbers. Even though these authors obtain a quite general result, it cannot be applied to games where agents' strategies are vector valued. For example when agents interact in networks, they have to decide whether to form bilateral

[4]The non cooperative formation of networks has been studied in the literature by Bala and Goyal (2000), however, their approach is very different from ours, because the non cooperative notion in that paper is related with the possibility of establishing links unilaterally, without the agreement of the partner. On the contrary, our model comes from the tradition that the existence of a link requires both parties to agree.

[5]For an excellent description and survey of the ensuing literature see Morris and Shin (2002).

relationships, so they have to make use of a vector valued set of action.[6]

In this sense, our paper extends an important uniqueness result in the global games literature to a more general class of games, games with vector valued space of actions where each component of the agents' strategy vector represents a binary decision. Even though the application to a link formation game is very natural, our extension can be applied to other problems beyond the network literature. Therefore, in particular, this binary decision can be seen as player's *intention* of establishing a link with other player, thus a link will be formed if and only if both players want to form the link.

We study a general class of games where the link formation process follows the strategic form of Dutta, van den Nouweland, and Tijs (1998),[7] such that if the payoff is parametrized by $x$, our main assumptions are: 1. *Increasing Differences:* player $i$'s incentive to choose a *higher* action is non decreasing in the others players' action profile. 2. *Proportional Incentive to Deviate:* player $i$'s incentive to deviate from the lowest possible vector of actions depends on the number of links requested by the player but not on the specific deviation. *3.* Existence of *upper* and *lower dominance regions*: for sufficiently low (high) values of the parameter, the action vector that represents link intention with nobody (everybody) is strictly dominant.

Under these assumptions, and some technical requirements, we prove that there exists a unique equilibrium profile surviving iterated elimination of strictly dominated strategies. The profile selected is independent of the noise size. The equilibrium strategy defines a unique $k^*$ such that $\forall x_i < k^*$ each player chooses the action vector

[6]It is important to note that the global game structure allows either for games with private or common values, ie. the payoff either depends on $x_i$ or $x$. The Carlsson and van Damme (1993) results are independent of this structure because of the simplicity of the interaction space ($2 \times 2$ game), but when we study a more general situation (more players or actions) the result may depend on the payoff structure. For details see Morris and Shin (2002).

[7]The strategic form approach of the link formation game was first proposed by Myerson (1991). The idea is that each player selects a list of the other players he wants to form a link with. Then the lists are put together and if the link $ij$ is required by both parts, then it is formed.

showing link intention with no other players, and $\forall x_i > k^*$ each player chooses the action vector showing link intention with all the other players.

In the limit when $\sigma \to 0$ the selected Bayesian Nash profile is in conflict with those arising from the application of traditional cooperative refinements of the network literature: SNE, CPNE and PS. This difference shows that the cooperative notions of stability are not robust to incomplete information in the form we introduce it. Moreover, we show that the stability notions based on cooperative refinements do not conflict with *efficiency* in our class of payoff functions, however, the equilibrium selected under the global games approach does conflict. These differences raise some practical questions about which criteria should be satisfied by networks that form in reality.

>From an applied point of view, the paper highlights the importance of two standard assumptions in the link formation literature. First, the assumption of complete information can be the origin of the multiplicity of networks supported by Nash Equilibria in link formation games. This multiplicity disappears when we perturb the game introducing incomplete information. Second, the cooperative refinements have been used to both, refine the multiplicity of equilibria in a link formation game and to argue that an existing network is stable to some cooperative deviations. However, the possibility of cooperation among coalitions of agents seems to be a more demanding assumption when the network is in formation than when the agents are maintaining or modifying an existing network. These observations raise some doubts about which is the pertinent equilibrium selection criteria in reality for a link formation game.

The paper is organized as follows. In section 2 we provide a simple example where we can show intuitively the main findings of the paper. Section 3 presents how we extend the global games literature generalizing the global games result to a class of games with vector valued action space. Section 4 contains the main results related to our class of link formation games. Here we develop the analysis using the cooperative refinements and our alternative approach to equilibrium selection, the global games results obtained in section 3. The main conclusions are contained in section 5. Finally, the proofs of propositions are relegated to the appendix.

## 2 An Illustrative Example

The idea of this section is to provide a simple example and an intuitive explanation of one of the main results developed in the paper. We are not going to be formal and technical details are postponed to next sections.

Consider a static link formation game of complete information $G$ with three players, where the set of strategies for each player $i$ is given by $A_i = \{0,1\}^2$. A strategy for player $i$ is a two component column vector of zeros and ones which identifies the set of players he wants to form links with. The players simultaneously choose strategies and a link between two players will be formed if and only if both of them want to form the link. For example, if the strategies of the players are $a_i = (a_{ij} = 1, a_{ik} = 1)$, $a_j = (a_{ji} = 0, a_{jk} = 1)$, $a_k = (a_{ki} = 0, a_{kj} = 1)$, then only the link $jk$ is created.[8] The payoff function for player $i$ is defined by:

$$\pi_i = a_{ij}a_{ji}\left(x + a_{jk}a_{kj}\beta x\right) + (\alpha x - c)a_{ij} + a_{ik}a_{ki}\left(x + a_{kj}a_{jk}\beta x\right) + (\alpha x - c)a_{ik} \quad (1)$$

The variable $x$ defines a level of profits which is assumed to be non negative, and $c$ is a fixed parameter that represents a level of investment incurred by agent $i$ for each link he wants to form. This investment is quasi specific to the partners, in the sense that if agent $i$ incurs an investment to agent $j$, then even if $j$ does not perform the reciprocal investment, and consequently the link $ij$ is not formed, agent $i$ receives

[8] Just for simplicity we wrote strategy $a_i$ as a row vector. In general we should see $a_i$ as column vector and any profile $a$ as a matrix. We will be more formal in the following sections.

a return $(\alpha x - c)$. The source of benefits $\alpha x$ is *independent* of other players' actions, in the sense that it can be obtained no matter the strategies the other players are following. On the other hand, if $j$ also performs the quasi specific investment to $i$ then the return to agent $i$ increases to $(x + \alpha x - c)$. In other words, there is an extra *direct* benefit $x$ from connection with each potential partner. Finally, agent $i$ profits from the relation between $j$ and $k$ when they are connected and provided that $i$ is connected with at least one of them. Note that if $i$ is connected with both of them, this *indirect* benefit is duplicated. For example, in the complete network the total payoff for player $i$ is given by $\pi_i = 2[x(1 + \alpha + \beta) - c]$. In this sense, $\beta x$ represent an indirect benefit or spillover that agent $i$ is able to extract from the connection between his partners and their partners. It seems natural to assume that $0 < \alpha < 1$, $0 < \beta < 1$, because we are scaling the benefits in relation to those obtained from reciprocity $(x)$.

One case where this kind of payoff function can be justified is in investment in R&D to reduce variable costs. In such a case, it has been empirically documented (see Goyal and Moraga-Gonzalez (2001)) that the firms tend to form alliances in pairs, represented by the links, but any reduction in cost obtained by $i$'s partners can be imitated by $i$, no matter if such reduction was obtained due to R&D of $i$'s partner or by a partner of $i$'s partners. We can assume that these firms are not competitors in any final market, so no negative externalities from R&D will arise.

### 2.1 The Nash Equilibria

Given the symmetry of the problem we are going to consider the best response correspondence for player 1. This correspondence, and the Nash equilibria arising, are different depending on the values of $x$. Figure 1 provides a summary of the different network structures supported by Nash equilibria, $NE(G)$, for different values of $x$.

Consider the following cases:

**Case (a):** Suppose that:

$$\frac{c}{1+\alpha} < x < \frac{c}{\alpha}$$

then the best response correspondence is given by:

$$BR_1(a_{-1}) = \begin{cases} a_{12} = 1, a_{13} = 1 & \text{if} \quad a_{21} = a_{31} = 1 \\ a_{12} = 1, a_{13} = 0 & \text{if} \quad a_{21} = 1, \; a_{31} = 0 \\ a_{12} = 0, a_{13} = 1 & \text{if} \quad a_{21} = 0, \; a_{31} = 1 \\ a_{12} = 0, a_{13} = 0 & \text{if} \quad a_{21} = 0, \; a_{31} = 0 \end{cases}$$

Note that, in this region, the strategies of agents 2 and 3 in relation to their connection does not affect the best response correspondence of agent 1. The intuition is that direct connections are enough to guarantee profitability. This characteristic leads to a multiplicity of Nash equilibria and, even more, it is possible to prove that all the feasible networks among the three agents can be supported by a Nash equilibrium.

**Case (b):** Suppose that:

$$\frac{c}{1+\alpha+\beta} < x < \frac{c}{1+\alpha}$$

then the best response correspondence is given by:

$$BR_1(a_{-1}) = \begin{cases} a_{12} = 1, a_{13} = 1 & \text{if} \quad a_{21} = a_{31} = a_{23} = a_{32} = 1 \\ a_{12} = 1, a_{13} = 0 & \text{if} \quad a_{21} = a_{23} = a_{32} = 1, \; a_{31} = 0 \\ a_{12} = 0, a_{13} = 1 & \text{if} \quad a_{21} = 0, \; a_{31} = a_{23} = a_{32} = 1 \\ a_{12} = 0, a_{13} = 0 & \text{otherwise} \end{cases}$$

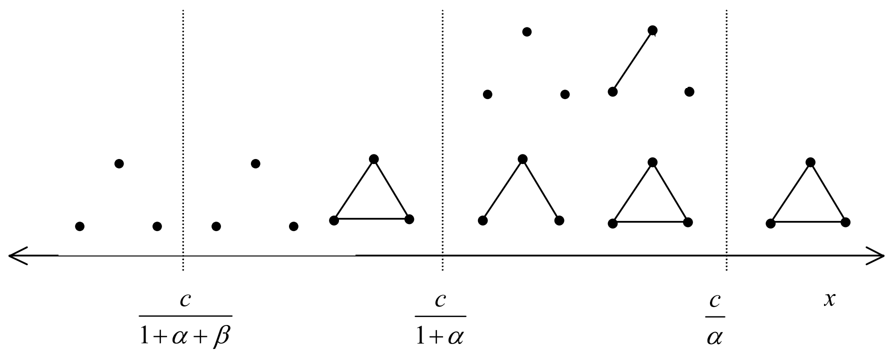


Figure 1: Network Structures supported by Nash Equilibria

Note that, in this case, the best response correspondence of player 1 is affected by the existence of the link between players 2 and 3. It is possible to prove that in this case only the empty and the complete network can be supported as a Nash equilibrium of the game.

**Case (c):** Finally, when $x < \underline{x} = c/(1+\alpha+\beta)$ a dominant strategy for any player $i$ is to play $a_i = (0,0) \equiv \mathbf{0}$. Analogously, when $x > \overline{x} = c/\alpha$ a dominant strategy is to form links with all the other players $a_i = (1,1) \equiv \mathbf{1}$, leading to the complete network.

### 2.2 Equilibrium Selection using Cooperative Refinements

A strategy profile is called a Strong Nash Equilibrium (SNE) if it is a Nash equilibrium and there is no coalition of players that can strictly increase the payoffs of all its members using a joint deviation (Aumann (1959)). On the other hand, a strategy profile is called a Coalition Proof Nash Equilibrium (CPNE) if, as in an SNE, no coalition can deviate to a profile that strictly improves the payoffs of all the players in the coalition. However, in the CPNE the set of admissible deviations is smaller, because the deviation has to be stable with respect to further deviations by subcoalitions. Finally, a network is Pairwise Stable (PS) if no pair of agents has incentives to form or sever one link.[9]

The application of these cooperative refinements to our three players game $G$ is very direct and a summary of results for the network supported by $SNE(G)$, $CPNE(G)$ and $PS(G)$ is given in figure 2.

First, it is possible to prove that, in this particular example, $SNE(G)$ coincides with $CPNE(G)$. Second, the analysis has to be performed in separated areas. It is easy to see that the strategy profile $a = (\mathbf{0}, \mathbf{0}, \mathbf{0}) \equiv [\mathbf{0}]$ is a $SNE(G)$ when $x < c/(1+\alpha+\beta)$, because for this range of values each agent plays $\mathbf{0}$ as a dominant strategy and, consequently, no coalition of agents can improve upon.[10] On the other hand, $a = (\mathbf{1}, \mathbf{1}, \mathbf{1}) \equiv [\mathbf{1}]$ is a $SNE(G)$ when $c/(1 + \alpha + \beta) < x$. The intuition is that the grand coalition playing $a_i = a_j = a_k = \mathbf{1}$ (which is a Nash equilibrium) can improve upon any other strategy profile (Nash equilibrium or not) given the complementarities involved in the payoff functions and the fact that a positive payoff is guaranteed.

We have to be more careful in the analysis of pairwise stability. If $c/(1+\alpha+\beta) < x < c/(1 + \alpha)$ then the empty and the complete networks are pairwise stable and consequently, pairwise stability does not refine the set of Nash equilibria. This result is a consequence that for these low values of $x$ the indirect connections are needed to make any connection profitable so, if nobody is making links, an agreement of two players to form a link is not enough to obtain a profitable relationship. On the other hand, if everybody is making links then no pair of agents benefits from severing a link. When $c/(1+\alpha) < x < c/\alpha$ any pair of agents which are not connected can profitably make a link and, consequently, only the complete network is pairwise stable. Finally,

[9] This concept has been defined directly over networks instead of strategy profiles (see Jackson and Wolinsky (1996)). In what follows we constraint the application of it to networks already supported by a Nash equilibrium. In this sense we consider Pairwise Stability as a refinement of the set of Nash equilibria. See the Appendix for a formal discussion.

[10] In what follows $[\mathbf{0}]$ and $[\mathbf{1}]$ represents a matrix full of zeros or ones respectively. The dimensionality is given by the profile they are representing. For example, in this three players case, $a = [\mathbf{0}]$ is a 2x3 matrix of zeros representing a complete strategy profile and $a_{-i} = [\mathbf{0}]$ is a 2x2 matrix of zeros representing a strategy profile that excludes player i's strategy.

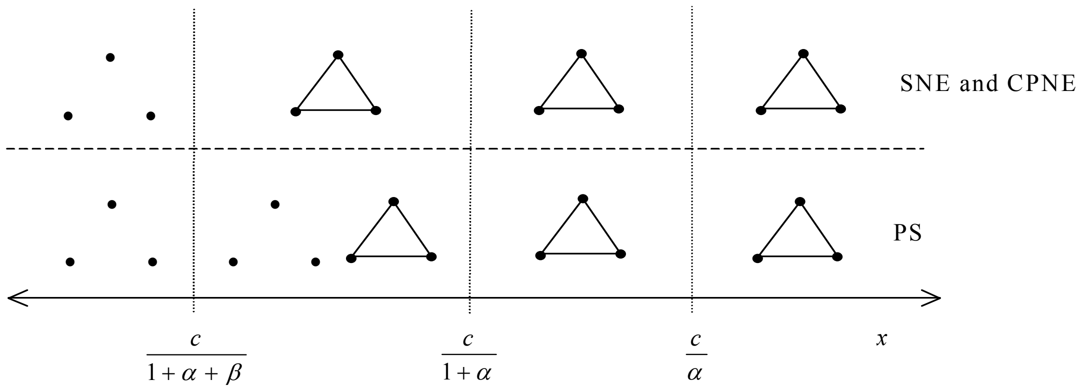


Figure 2: Network structures supported by Strong Nash Equilibrium (SNE), Coalition Proof Nash Equilibrium (CPNE) and Pairwise Stability (PS).

if $x < c/(1+\alpha+\beta)$ then action $a_i = \mathbf{0}$ is a dominant strategy for player $i$ and the unique pairwise stable network is the empty one. Analogously, if $x > c/\alpha$ then action $a_i = \mathbf{1}$ is a dominant strategy and the unique pairwise stable network is the complete one.

### 2.3 Equilibrium Selection using the Global Games Approach

Suppose now we allow some arbitrary amount of incomplete information in the payoff structure such that player $i$'s payoff function depends on a private value $x_i$, which is observed by $i$ and contains diffuse information about $x$. The private value has the following structure: $x_i = x + \sigma\varepsilon_i$, where $\sigma > 0$ is a scale factor, $x$ is drawn from $\left[\underline{X}, \overline{X}\right]$ with uniform density and $\varepsilon_i$ is an independent realization of the density $\phi$ with support in $[-\frac{1}{2}, \frac{1}{2}]$. We assume $\varepsilon_i$ is *i.i.d.* across the individuals.[11]

In this context of incomplete information, a Bayesian pure strategy for player $i$ is a function $s_i : [\underline{X} - \frac{\sigma}{2}, \overline{X} + \frac{\sigma}{2}] \rightarrow A_i$, and $s = (s_1, s_2, s_3)$ is a pure strategy profile, where $s_i \in S_i$. Calling this game of incomplete information $G(\sigma)$, let us define as

[11] Note that $\phi$ need not be symmetric around the mean nor even have zero mean. Also note that $x_i$ contains diffuse information about player $j$'s private value: $x_j$.

$BNE(G(\sigma))$ the set of Bayesian Nash equilibria of $G(\sigma)$.

**Proposition 1**: $\forall\ \sigma > 0$ *there exists a unique strategy profile* $s^*$, *that survives iterated elimination of strictly dominated strategies, where:*

$$s_i^*(x_i) = \begin{cases} \mathbf{1} & if \quad x_i > k^* \\ \mathbf{0} & if \quad x_i < k^* \end{cases} \qquad \forall i \ \ and\ k^* = \frac{4c}{2+4\alpha+\beta}$$

Since the noise structure is $x_i = x + \sigma\varepsilon_i$, as $\sigma \to 0 \quad x_i \to x$, thus the unique equilibrium selected implies that $\forall\ x < k^*$ all agents play the action **0**, so the empty network is formed, and $\forall\ x > k^*$ all the agents playing action **1** and, consequently, the complete network is formed. Conditional on the private value, figure 3 shows the networks supported by this equilibrium as the noise goes to zero.

This proposition is a particular case of proposition 2, so we are not going to give a formal proof here. Instead, we are going to discuss the intuition behind the proposition. Consider players 2 and 3 using any strategy. It is common knowledge of the game that these strategies must consider playing the actions **0** and **1** in the previously identified dominance regions. It is possible to prove that agent 1's best response to such strategies is a strategy that considers playing **0** when $x_1 < \underline{x}^1$ and playing **1** when $x_1 > \overline{x}^1$ where $\underline{x} < \underline{x}^1$ and $\overline{x} > \overline{x}^1$. In other words, in equilibrium, the regions where **0** and **1** are played has been extended. Given the symmetry of the problem, all the agents perform the same analysis and consequently the regions where **0** and **1** are played are extended symmetrically for all the players. Iterating with this argument, it is possible to generate increasing and decreasing sequences $\{\underline{x}^n\}_{n=1}^{\infty}$ and $\{\overline{x}^n\}_{n=1}^{\infty}$, respectively, such that they have the same limit value, i.e., $\underline{x}^{\infty} = \overline{x}^{\infty} \equiv k^*$.

Finally, it is important to notice that the equilibrium profile selected in $G(\sigma)$ does not depend on the size of the noise. In this sense, we say that $s^*$ is the unique

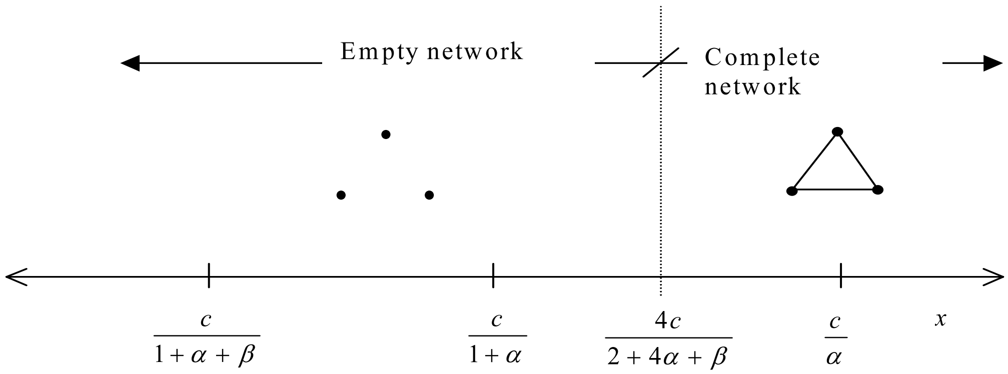


Figure 3: Equilibrium Selected using the Global Game Approach

equilibrium of the link formation game $G$, which is stable to the way how we introduce incomplete information in the parameter $x$.

### 2.4 Efficient Allocation of the Game

The efficient allocation of the game $E(G)$, is defined for each $x$ as the strategy profile that maximizes the sum of the payoffs for the players. It is easy to check that:

$$E(G) = \begin{cases} \{[\mathbf{0}]\} & \text{if } x < c/(1+\alpha+\beta) \\ \{[\mathbf{1}]\} & \text{if } x > c/(1+\alpha+\beta) \end{cases}$$

and as a result, the networks supported by efficient allocation $E(G)$ coincide with those supported by the sets $SNE(G)$ and $CPNE(G)$ described in figure 2.

### 2.5 Discussion

The example developed illustrate the main results of the paper. First, the traditional cooperative refinements do not conflict with the efficient allocation for each level of $x$. However, the equilibrium selected using the global games approach clearly conflicts with efficiency when $\frac{c}{1+\alpha+\beta} < x < \frac{4c}{2+4\alpha+\beta}$. Second, in the interval $\frac{c}{1+\alpha} < x < \frac{4c}{2+4\alpha+\beta}$

all the cooperative refinements predict the formation of the complete network, however, our selected equilibrium predicts the empty network. This means that, for these values of $x$, it is impossible to satisfy the two stability conditions simultaneously. Giving that each stability notion leads to the selection of a different equilibrium, we have two implications. First, the cooperative refinements are not robust to our incomplete information structure. Second, the feasibility of the equilibrium selected by each approach depends critically on the feasibility of the deviations considered in each stability condition. For example, if the game presents a coordination problem, then it does not seem reasonable to select the equilibrium using cooperative refinements, which ignore the coordination problem itself. In such a case, the global games approach could be more adequate. In the particular case of our link formation game (previously studied by Dutta, van den Nouweland and Tijs (1998)), the game is essentially non cooperative[12] and there exists a coordination problem. If, in addition, we consider a supermodular payoff function as in (1), then the cooperative refinements should not conflict with efficiency, because the agents can coordinate actions in the grand coalition, leading to the efficient allocation of the game.

In what follows we are going to show that these findings hold in a much more general setting. We will keep the strategic form approach proposed by Myerson (1991) and studied by Dutta et al. (1998), but we extend our results to a family of payoff functions that satisfies a set of assumptions. Our first step will be to expand the applicability of Global Games theory in order to use it in a class of link formation games.

[12] We do not see any reason to assume in advance that cooperation is possible.

## 3 Global games with strategic complementarities and vector valued set of actions

In this section we extend the global games literature generalizing the global games result to a class of games with vector valued action space. We present a more general set up than a link formation game, where the dimension of the strategy vector does not coincide necessarily with the number of opponents.[13] Using standard assumptions of the global games literature and an additional specific assumption about the vector valued strategies we extend the literature finding a uniqueness result: the class of games analyzed have a unique equilibrium strategy profile obtained through iterated elimination of strictly dominated strategies. Later, in section 4, we will use this result and other refinements to analyze in detail the special case of link formation games.

Let us consider the following general setup for a $I$ person game $\Gamma$. There exists $I$ players indexed by $i$, each player has a $N-$dimensional set of strategies $A_i = \{0,1\}^N$.[14] Then a strategy $a_i$ for player $i$ is a column vector of zeros and ones, i.e. $a_i \in A_i$, and an element of $a_i$ is $a_{ki} \in \{0,1\}$ where $i \in \{1,...,I\}$and $k \in \{1,...,N\}$.

For simplicity we will assume symmetric players with a payoff function given by $\pi(a_i, a_{-i}, x)$ where $a_i \in A_i$, $a_{-i} \in A_{-i} = \times_{j \neq i} A_j$,[15] and $x \in [\underline{X}, \overline{X}] \subset \mathbb{R}$ is an exogenous variable. We define $\Delta\pi(a_i, a_i'; a_{-i}, x) = \pi(a_i, a_{-i}, x) - \pi(a_i', a_{-i}, x)$ as agent

[13] In a general class of link formation games, players have to decide whether to connect, or to form a link, with each of his "opponents". Under this view, naturally each player's strategy is a vector, where each component is a binary decision and the vector dimension corresponds to the number of his opponents.

[14] Note that when $N = I - 1$, we can interpret $\Gamma$ as a link formation game, where every player has to decide whether to form a link with the rest of the players. Naturally, each player has $I-1$ binary decisions to make, so the player's strategy is a vector of $I-1$ dimension.

[15] $A_i$ is a partially ordered set:
$a_i \geq \widehat{a}_i$ if $a_{ji} \geq \widehat{a}_{ji}$ $\forall j = 1...N$
$a_i > \widehat{a}_i$ if $a_{ji} \geq \widehat{a}_{ji}$ $\forall j = 1...N$ and $a_{ji} > \widehat{a}_{ji}$ for some $j$.
In the same way $A_{-i}$ is a partially ordered set:
$a_{-i} \geq \widehat{a}_{-i}$ if $\forall j \neq i$ $a_j \geq \widehat{a}_j$
$a_{-i} > \widehat{a}_{-i}$ if $\forall j \neq i$ $a_j \geq \widehat{a}_j$ and $a_j > \widehat{a}_j$ for some $j$.

$i$'s payoff difference when he changes from action $a_i'$ to action $a_i$. We also define $A = A_i \times A_{-i}$.

Define $A_i^n \subset A_i$ such that, if $a_i \in A_i^n$ then $a_i$ is $N$-dimensional column vector which contain $n$ components 1 and $N-n$ components 0. In fact, the family of sets $\{A_i^n\}_{n=0}^N$ defines a partition of $A_i$ because $A_i = \bigcup_{n=0}^{N} A_i^n$ and $A_i^n \cap A_i^{n'} = \emptyset$ for all $n$ and $n' \in \{0...N\}$ with $n \neq n'$. Additionally, it is easy to see that $A_i^N$ and $A_i^0$ are singleton, therefore if $a_i \in A_i^N$ then $a_i$ is a column of ones, which we denote as $a_i \equiv \mathbf{1}$ and it is defined as the *highest* action vector. Equivalently, if $a_i \in A_i^0$ then $a_i$ is a column of zeros, denoted by $a_i \equiv \mathbf{0}$ and it is defined as the *lowest* action vector. If $a_i \in \{\mathbf{1}, \mathbf{0}\} \subset A_i$, then $a_i$ is an *homogenous* action vector, and $A_i^h \equiv A_i^0 \cup A_i^N = \{\mathbf{1}, \mathbf{0}\}$ is defined as players $i$' set of homogenous actions.

Similarly consider $A_{-i}^h = \times_{j \neq i} A_j^h$ and define $A_{-i}^{h,l}$ such that if $M_l$ is an element of $A_{-i}^{h,l}$, then $M_l$ is $N \times (I-1)$ matrix containing $l$ columns $\mathbf{1}$ and $(I-1)-l$ columns $\mathbf{0}$. As we consider above, the family of sets $\left\{A_{-i}^{h,l}\right\}_{l=0}^{I-1}$ is a partition of $A_{-i}^h$ because $A_{-i}^h = \bigcup_{l=0}^{I-1} A_{-i}^{h,l}$ and $A_{-i}^{h,l} \cap A_{-i}^{h,l'} = \emptyset$ for all $l$ and $l' \in \{0...I-1\}$ with $l \neq l'$. In particular $M_{I-1} = [\mathbf{1}]$ is a $N \times (I-1)$ matrix of ones, and $M_0 = [\mathbf{0}]$ is a $N \times (I-1)$ matrix of zeros.

Let us consider the following assumptions in the payoff structure:

**(A1). Increasing Differences (ID).** Conditional on the value of the exogenous parameter $x$, the greater the other players' action profile the greater is player $i$'s incentive to choose a higher action:

$\forall a_i \in A_i$ and $\forall a_{-i} \in A_{-i}$

If $a_i \geq a_i'$ and $a_{-i} \geq a_{-i}'$, $\Delta\pi(a_i, a_i'; a_{-i}, x) \geq \Delta\pi(a_i, a_i'; a_{-i}', x)\ \forall x$

Moreover:

**a.** $\forall a_i \neq \mathbf{0}$, $\exists a_{-i} \neq [\mathbf{1}] \in A_{-i}^h$ *s.t.*:

$\Delta\pi(a_i, \mathbf{0}; a_{-i}, x) < \Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x)\ \forall x$

**b**. $\forall a_i \neq \mathbf{1}$, $\exists a_{-i} \neq [\mathbf{0}] \in A^h_{-i}$ $s.t.$:

$\Delta\pi(\mathbf{1}, a_i; a_{-i}, x) > \Delta\pi(\mathbf{1}, a_i; [\mathbf{0}], x)\ \forall x$

**(A2). Continuity (C).**

$\pi(a_i, a_{-i}, x)$ is a continuous function in $x$

**(A3). Monotonicity (M).** The greater the value of the exogenous parameter $x$, the greater is player $i$'s incentive to choose a higher action:

$\exists\ c > 0\ s.t.\ \forall\, a_i > a_i' \,\forall\, a_{-i}$ and $x,\, x' \in [\underline{X}, \overline{X}]\ \ x > x'$

$\Delta\pi(a_i, a_i'; a_{-i}, x) - \Delta\pi(a_i, a_i'; a_{-i}, x') > c\, \|a_i - a_i'\|\, (x - x')$

**(A4). Proportional Incentive to Deviate (PID).** If the players other than $i$ are randomizing with equal probability among homogeneous actions such that a fixed number of highest actions is played, then the (expected) value of the incentive of agent $i$ to deviate from the homogenous action $\mathbf{0}$ to any other action $a_i$ varies proportionally with the *elemental* deviation, the deviation to any strategy that has just one component equal to 1:

$\exists\ \lambda : A_i \rightarrow [0, \infty)$ satisfying $\lambda(\mathbf{0}) = 0$ and $\lambda(a_i) = 1\ \ \forall a_i \in A_i^1$, s.t. if $a_i \in A_i^n$ and $a_i' \in A_i^{n'}$, then:

$\lambda(a_i) > \lambda(a_i') \Leftrightarrow n > n'$,

$\lambda(a_i) = \lambda(a_i') \Leftrightarrow n = n'$

And $\forall a_i \in A_i$

$\sum_{a_{-i} \in\ A^{h,l}_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) = \lambda(a_i) \sum_{a_{-i} \in\ A^{h,l}_{-i}} \Delta\pi(a_i', \mathbf{0}; a_{-i}, x)\ \forall l = 0, ..., I-1\ \ \forall a_i' \in A_i^1$

**(A5). Upper and Lower Indifference Values (IV).** If all other players are choosing the highest (lowest) action, there exists a unique value of $x$ such that player $i$ is indifferent between the lowest (highest) action and any other action.

$\forall a_i \in A_i \backslash \mathbf{0}$, $\exists!\ \underline{x} > \underline{X}\ \ s.t.\ \ \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$, and

$$\forall a_i \in A_i \backslash \mathbf{1},\ \exists!\ \overline{x}\ \ s.t.\ \overline{X} > \overline{x} > \underline{x}\ \text{ and }\ \Delta\pi(\mathbf{1}, a_i; a_{-i} = [\mathbf{0}], \overline{x}) = 0.$$

Assumption 5 says that for each $a_i \in A_i$ the values $\underline{x}$ and $\overline{x}$ exist and they are unique. In principle, however, they could depend on $a_i$. The following lemma shows that this is not the case.

**Lemma 1**: *The indiference values $\overline{x}$ and $\underline{x}$ in assumption 5 are independent of $a_i \in A_i$.*

An aditional important remark is that assumptions A1 (ID), A3 (M) and A5 (IV) provide sufficient conditions for the existence of dominance regions, along which each action is strictly dominant, providing this setup with the necessary global games structure. i.e.

$\forall\, x < \underline{x}$, $\Delta\pi(a_i, \mathbf{0}; a_{-i}, x) < 0\ \ \forall a \in A$, and

$\forall\, x > \overline{x}$, $\Delta\pi(\mathbf{1}, a_i; a_{-i}, x) > 0\ \ \forall a \in A$

Suppose now that the game is one of incomplete information in the payoff structure. Player $i$' payoff function depends now on a private value $x_i$ instead of $x$. These values are, however, related so that $x_i$ also constitutes a noisy signal of $x$ observed by player $i$. [16]

Each player' value has the following structure: $x_i = x + \sigma\varepsilon_i$ ,where $\sigma > 0$ is a scale factor, $x$ is drawn from the interval $[\underline{X}, \overline{X}]$ with uniform density, and $\varepsilon_i$ is a random variable distributed according to a continuous density $\phi$ with support in the interval $[-\frac{1}{2}, \frac{1}{2}]$. We assume $\varepsilon_i$ is *i.i.d.* across the individuals.

This general noise structure has been used in the global games literature, allowing us to model in a simple way the conditional distribution of the opponents value given

[16] It is possible to model the private values case as a limit of the common value case (when players derive utility from the actual value of the variable $x$ and the $x_i$'s are pure signals of $x$) as the noise goes to zero ($\sigma \to 0$). We do not pursue this approach in our paper but it has been used in the global game literature (Carlsson and van Damme (1993), Morris and Shin (2002) and Frankel, Morris and Pauzner (2002)).

a player's own value. The conditional distribution of an opponent's value $x_j$ given the own value $x_i$ admits a continuous density $f_\sigma$ and a cdf $F_\sigma$ with support in the interval $[x_i - \sigma, x_i + \sigma]$. Moreover this literature establishes a significant result: when the prior is uniform, players' posterior beliefs about the difference between their own value and other players' values are the same, i.e. $F_\sigma(x_i \mid x_j) = 1 - F_\sigma(x_j \mid x_i)$.[17]

In this context of incomplete information, a Bayesian pure strategy for a player $i$ is a function $s_i : [\underline{X} - \frac{\sigma}{2}, \overline{X} + \frac{\sigma}{2}] \to A_i$, and $s = (s_1, s_2, ..., s_I)$ is a pure strategy profile, where $s_i \in S_i$. Equivalently we define $s_{-i} = (s_1, s_2, ..s_{i-1}, s_{i+1}, ...s_I) \in S_{-i}$.

In particular, a switching strategy between the lowest and the highest action is a Bayesian pure strategy satisfying : $\exists\, k_i$ $s.t.$

$$s_i(x_i) = \begin{cases} \mathbf{1} & if \quad x_i > k_i \\ \mathbf{0} & if \quad x_i < k_i \end{cases}$$

Abusing notation, we write $s_i(\cdot; k_i)$ to denote the switching strategy with threshold $k_i$.

Finally, if player $i$ has a private value $x_i$ and facing a strategy $s_{-i}$ his expected payoff can be written as

$$\Pi_i(a_i, s_{-i}, x_i \mid x_i) = \int_{x_{-i}} \pi(a_i, s_{-i}(x_{-i}), x_i) dF_{\sigma(-i)}(x_{-i} \mid x_i)$$

Calling this game of incomplete information $\Gamma(\sigma)$, let us define $BNE(\Gamma(\sigma))$ as the set of Bayesian Nash equilibria of $\Gamma(\sigma)$. In addition, we assume:

**(A6). Single Crossing (SC).** There exists a unique value $k^*$, of the exogenous variable such that if player $i$ has a value $x_i = k^*$ and he believes that all other players are using a switching strategy between **0** and **1** with threshold $k^*$, the expected value

[17] This property holds approximately when $x$ is not distributed with uniform density but $\sigma$ is small, i.e. $F(x_i \mid x_j) \approx 1 - F(x_j \mid x_i)$. See details in Lemma 4.1 Carlsson and van Damme (1993).

of his payoffs when he chooses **0** or **1** are the same, i.e.:

$$\text{There exists a unique } k^* \text{ solving } \sum_{n=0}^{I-1} \sum_{a_{-i} \in A_{-i}^{h,n}} \{\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, k^*)\} = 0$$

One of the main results of the paper proves that $\Gamma(\sigma)$ has a unique profile $s^*$, played in equilibrium $\forall \sigma > 0$, and in this profile every player will play a switching strategy $s_i(\cdot; k^*)$ with $k^*$ according A6 (SC).

**Proposition 2**: *Consider the game $\Gamma(\sigma)$. Under assumptions A1 to A6:*

$\forall\ \sigma > 0$ *there exists a unique strategy profile $s^*$ surviving iterated elimination of strictly dominated strategies, where:*

$$s_i^*(x_i; k^*) = \begin{cases} \mathbf{1} & if \quad x_i > k^* \\ \mathbf{0} & if \quad x_i < k^* \end{cases} \qquad \forall i \ \ and \ \ \underline{x} < \ k^* < \overline{x}$$

This result extends the global games literature. With Proposition 2 we make available an important uniqueness result to a more general class of games with vector valued space of actions. For example, we could study the foreign investment decisions of $I$ investors in $N$ different countries in the presence of investment complementarities or the decision to enter in $N$ new research areas by $I$ researchers in the presence of synergies. In this paper we will use the result to analyze a link formation game where each player decides to request or not a link with the other players and then the dimension of the action space is equal to the number of other players, i.e. $N = I - 1$.

## 4 Stability and Equilibrium Selection in a Class of Link Formation Games

In this section we go back to the link formation game in strategic form and we refine the set of Nash equilibria using different equilibrium selection approaches. In subsection 4.1 we describe the general link formation game under analysis, in particular, we discuss the assumptions required over the payoff structure for applying our global games generalization developed in section 3. In subsection 4.2 we apply cooperative refinements to the game while in 4.3 we use our equilibrium selection approach. In 4.4 we briefly discuss an application.

Several authors have studied the theoretical foundations of network formation and its properties (Myerson (1977), Aumann and Myerson (1988), Dutta, van den Nouweland and Tijs (1998), Slikker and van den Nouweland (2000) among others). Particular emphasis has been given to study the link formation process and the conflict between stability and efficiency in networks (Jackson and Wolinsky (1996), Dutta and Mutuswami (1997)). The link formation literature precedes the stability/efficiency literature, however, the insights from the latter area have interacted and motivated more research in the former.

In this paper we focus on the link formation game in strategic form of Dutta, van den Nouweland and Tijs (1998) but we introduce a different, non cooperative, equilibrium selection approach. Our approach is in fact based on a different stability notion and consequently, we can analyze the properties of the selected equilibria and compare them with those obtained from the cooperative results. We also discuss the traditional stability/efficiency conflict when our stability notion is being used.

### 4.1 The Link Formation Game

We will consider a general class of link formation game $G$ having the same structure of the class of games analyzed in the previous section. In particular, since $G$ is a link formation game, the dimension of the strategy vector will coincide with the number of players minus one (the dimension of the strategy vector is equal to the number of opponents or $N = I - 1$).

For simplicity we assume that there exists $N+1$ players indexed by $i$ and each player has a set of strategies $A_i = \{0,1\}^N$ . In this context, a strategy for player $i$ is a column vector of zeros and ones which identify the set of players he wants to form links with. The players simultaneously choose strategies and a link between two of them will be formed if and only if both players want to form the link. For example, if players' strategies are such that $a_i' = (..., a_{ji} = 1, a_{ki} = 1, ...)$, $a_j' = (..., a_{ij} = 0, a_{kj} = 1, ...)$, $a_k' = (..., a_{ik} = 0, a_{jk} = 1, ...)$, then the link $jk$ is created.[18]

We assume symmetric players with a payoff function given by $\pi(a_i, a_{-i}, x)$ where $a_i \in A_i$, $a_{-i} \in A_{-i} = \times_{j \neq i} A_j$, and $x \in [\underline{X}, \overline{X}] \subset \mathbb{R}$ is an exogenous variable. We define $\Delta\pi(a_i, a_i'; a_{-i}, x) = \pi(a_i, a_{-i}, x) - \pi(a_i', a_{-i}, x)$ as agent $i$'s payoff difference when he changes from action $a_i'$ to action $a_i$.

The set of assumptions over the payoff function were formally defined in section 3. In the particular case of a link formation game, the assumptions can be interpreted using a network language as follows:

**(A1). Increasing Differences (ID).** The more connected is the network formed by players other than $i$, the higher the incentive of player $i$ to choose a vector of actions that permit him to be more connected with the rest of the network. There exists scenarios where the incentive is strict, for example, the incentive could be strict if the strategies of the players implies an increase in the number of direct links between player $i$ and the rest of the network.

**(A2). Continuity (C).** The payoff function is continuous in the parameter of benefits $x$.

[18] We consider the strategy $a_i$ as a $N$ dimension column vector. In what follows we identify the $k$ component of $a_i$ with the link intention of player $i$ toward player $k$ and we denote it $a_{ki}$ which is a slightly different notation from the one used in section 2 (we inverted the order of $k$ and $i$). We need this notation because we are going to attach different column vectors to form strategy profiles and we want to use a more standard matricial notation. In addition, in this interpretation $a_{ii}$ does not play any role and then it is ommited from $a_i$.

**(A3). Monotonicity (M).** Conditional on the strategies played by other agents, the greater the benefits of being connected the highest the incentive for player $i$ to request the formation of links with other players.

**(A4). Proportional Incentive to Deviate (PID).** If the players other than $i$ are randomizing, with equal probabilities, among homogenous actions such that a fixed number of them is connected, then the (expected) value of the incentive for player $i$ to deviate from a homogenous action $\mathbf{0}$ to any other action $a_i$ is proportional to the (expected) value of the incentive to deviate to an action involving just one link intention.[19] The proportionality coefficient increases in the number of links requested by player $i$ in the strategy $a_i$.

**(A5). Upper and Lower Indifference Values (IV).** Consider a given action $a_i$ for player $i$. There exists a sufficiently low value for the parameter of benefit such that, even if all the other players are requesting links with everybody else, player $i$ is indifferent between actions $a_i$ and $\mathbf{0}$. Analogously, there exists a sufficiently high value for the parameter of benefit such that, even if all the other players are not requesting links at all, player $i$ is indifferent between actions $a_i$ and $\mathbf{1}$.

### 4.2 Equilibrium Selection using Cooperative Refinements

In this section we are interested in applying some of the most commonly used stability concepts to our problem in order to refine the multiplicity of Nash equilibria that can arise in our game. Three concepts have been proposed: *Pairwise Stable Nash Equilibrium (PS)*, *Coalition Proof Nash Equilibrium* (CPNE) and *Strong Nash Equilibrium* (SNE). A formal definition of the first concept is in the Appendix, while the other two can be revised in Dutta and Mutuswami (1997).

[19] This notion can be related to the idea of *anonimity* in the network literature. The identity of the players with whom agent $i$ is requesting a link in $a_i$ does not matter. The number of requests, however, is important.

It has been proved that,[20] for a general link formation game $\Gamma$ under complete information[21]:

$$\begin{aligned} SNE(\Gamma) &\subseteq PS(\Gamma) \subseteq NE(\Gamma) \\ SNE(\Gamma) &\subseteq CPNE(\Gamma) \subseteq NE(\Gamma) \end{aligned} \tag{2}$$

On the other hand, the set of *Efficient Allocations* of a game $\Gamma$, $E(\Gamma)$, is defined as:

$$E(\Gamma) = \{a^* \in A,\ \textit{such that } a^* \in \arg\max_{a} \sum_{i \in \mathcal{N}} u_i(a)\} \tag{3}$$

An important implication of the theoretical conflict between efficiency and stability is that, in general, $E(\Gamma) \nsubseteq PS(\Gamma)$ and $E(\Gamma) \nsubseteq CPNE(\Gamma)$.[22] We are going to show that this is not the case in our game.

Consider the link formation game in strategic form $G$ defined at the begining of section 4 and satisfying the assumptions A1 to A5. In addition, we introduce the following assumption:

**(A7). Status Quo Payoff (SQP)**

$\pi(\mathbf{0}, a_{-i}, x) = 0 \quad \forall a_{-i} \in A_{-i}\ ,\ \forall x \in [\underline{X}, \overline{X}]$

This assumption is very natural in the sense that if agent $i$ does not require any link, then he has neither benefits nor costs. The role of assumption A7 (SQP) is to permit us to write $\Delta\pi(a_i, \mathbf{0}; a_{-i}, x) = \pi(a_i, a_{-i}, x)$ so that all the assumptions over $\Delta\pi$ can be directly interpreted in terms of $\pi$.

[20] See Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997).

[21] Even when Pairwise Stability has been defined over graphs, we can talk about the set $PS(\Gamma)$ as the subset of Nash Equilibria leading to the formation of pairwise stable graphs. See the Appendix for a formal definition.

[22] See Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997), respectively.

Under this general set of assumptions it is not easy to give a detailed description of the set of Nash equilibria of the game $G$. However, it is possible to show that some particular profiles are indeed Nash Equilibria, and even more, we can show that these equilibria are stable under the traditional cooperative refinements. In addition, we will show that the set $E(G)$ is always stable under the different cooperative notions.

**Proposition 3**: *Consider the link formation game $G$. Under assumptions A1 to A5 and A7 we have:*

a. *The set $E(G)$, satisfies:*

$$E(G) = \begin{cases} \{[\mathbf{0}]\} & \text{if } x < \underline{x} \\ \{[\mathbf{1}]\} & \text{if } x > \underline{x} \end{cases}$$

b. *If $a \in E(G)$ then $a$ is stable under all the cooperative refinements.*

Proposition 3 shows that in the class of supermodular games defined in section 4 under assumptions A1 to A5 and A7 there is no conflict between efficiency and the cooperative notions of stability.

### 4.3 Equilibrium Selection using the Global Games Approach

Now we follow the global games approach to study equilibrium selection process in $G$. Basically, we introduce some arbitrary amount of incomplete information in the payoff structure such that player $i$'s payoff function depends on a private value $x_i$, which is observed by $i$ and contains diffuse information about $x$. The value follows the standard global games structure, which was described in detail previosly. The interpretation of (A6) in the link formation game is the following:

**(A6). Single Crossing (SC).** If the players other than $i$ are randomizing among homogenous actions with equal probabilities then there exist a unique value for the parameter of benefits $k^*$, such that player $i$'s expected payoff from playing $\mathbf{0}$ (none link is requested) or $\mathbf{1}$ (all links are requested) is the same.

In this context of incomplete information we call the game $G(\sigma)$ and define as $BNE(G(\sigma))$ the set of Bayesian Nash equilibria of $G(\sigma)$.

Since in section 3 we have proved a general uniqueness result, we use our proposition 2 to show that $G(\sigma)$ have a unique equilibrium strategy profile.

**Corollary of proposition 2**: *Consider the link formation game* $G(\sigma)$. *Under assumptions A1 to A6:*

$\forall\ \sigma > 0$ *there exists a unique strategy profile* $s^*$ *surviving iterated elimination of strictly dominated strategies, where:*

$$s_i^*(x_i; k^*) = \begin{cases} \mathbf{1} & if \quad x_i > k^* \\ \mathbf{0} & if \quad x_i < k^* \end{cases} \qquad \forall i \;\; and \; \underline{x} < \; k^* < \overline{x}$$

The equilibrium strategy defines a unique $k^*$ according to assumption A6. This threshold satisfies that $\forall x_i < k^*$ each player chooses the action vector that shows link intention with no other players, and $\forall x_i > k^*$ each player choose the action vector that shows link intention with all the other players. It is important to notice that the equilibrium profile selected does not depend on the size of the noise $\sigma$, and it does not depend on the noise structure $\phi$ either. We have assumed that the parameter $x$ is distributed according to a flat prior, but it is possible to prove that any prior can be treated as a flat prior when $\sigma$ goes to zero. In this sense, we say that $s^*$ is the unique equilibrium of the link formation game $G$, which is robust to our incomplete information structure in the parameter $x$.

Even though the proposition proves that when $\sigma > 0$ each player is using a switching strategy $s_i^*$, the network formed depends on the size of the noise. In general, if some $x_i > \ k^* + \sigma$ then every player receives a value greater than $k^*$ and therefore the complete network is formed. Equivalently if some $\ x_i < \ k^* - \sigma$ the empty network is formed, but if all $x_i \in [k^* - \sigma, k^* + \sigma]$ then any network can be formed depending on the realization of each player's value. Following this analysis is easy to see that as $\sigma$ goes to zero just two possibilities remain, the complete and the empty network.

The corollary establishes a significant result in the link formation literature. Comparing Proposition 3 and the corollary, we can see that the actions played in $\lim_{\sigma \to 0} BNE(G(\sigma))$ are not efficient ($k^* > \underline{x}$) and thus there is a conflict between efficiency and stability, when the later is understood as stability to the introduction of incomplete information with our structure. This conflict does not arise when the cooperative stability notions are considered. More importantly, as the example in section 2 illustrates, cooperative and non cooperative notions of stability could not be satisfied simultaneously for some range of the parameter of profits $x$, and then we are forced to decide which approach is more pertinent to refine the set of Nash equilibria in link formation games. Given that the link formation game under analysis is non cooperative in nature and a coordination problem exists, it seems more reasonable to use a non cooperative equilibrium selection approach instead of assuming ex-ante the possibility of cooperation among agents.

### 4.4 Application

One question that need to be answered is how difficult is to check the set of assumptions in particular applications. In this subsection we consider an extension of the three players game discussed in section 2 and we show that all the assumptions can be checked directly.

Consider the link formation game $G$ with $N + 1$ player, such that each player has the same following payoff function:[23]

$$\pi_i(a_i, a_{-i}, x) = \sum_{j \neq i} \left\{ a_{ij} a_{ji} \left( x + \sum_{k \neq i \neq j} a_{jk} a_{kj} \beta x \right) + (\alpha x - c) a_{ji} \right\} \tag{4}$$

which is a generalization of the payoff function described in equation (1). The interpretation of the different components of this function (independent, direct and indirect benefits) is the same as in section 2.

It is clear that the game played is different depending on the values of $x$. In particular, when $x(1 + \alpha + \beta) < c$ a dominant strategy for any agent $i$ is to play $a_i = \mathbf{0} \quad \forall i \in \{1, ..., N+1\}$, forming the empty network. On the other hand, when $\alpha x > c$, then a dominant strategy is to play $a_i = \mathbf{1} \quad \forall i \in \{1, ..., N+1\}$, forming the complete network.

Assumptions A1 to A5 and A7 can be checked easily. Assumption A6 is verified in the following Lemma.[24]

**Lemma 2**: *Consider the link formation game G when the payoff function has been specialized according to* (4)*. The Single Crossing assumption A6 is satisfied with*:

$$k^* = \frac{Nc \sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix}}{\sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix} N\alpha + \left[ (1+\beta) \sum_{n=0}^{N} \left\{ \begin{pmatrix} N \\ n \end{pmatrix} n \right\} - N\beta \right]} \tag{5}$$

Lemma 2 permits us to apply the equilibrium selection by the global games approach to the payoff function defined by (4). The equilibrium selected generalizes the result discussed in section 2.

[23] We are using the notation introduced in section 3, which is slightly different from section 2. Under this notation $a_{ji}$ represents the link intention of agent $i$ with respect to agent $j$.

[24] The proof of Lemma 2 is available from the authors upon request.

## 5 Conclusion

The goal of this paper was to use a non cooperative equilibrium selection approach as a notion of stability in link formation games. Specifically, we studied the link formation game in strategic form of Dutta, van den Nouweland and Tijs (1998) where we constrained the payoffs to a class of supermodular functions defined by assumptions A1 to A5. Assumption A6 (SC) was introduced to apply the global games approach and assumption A7 (SQP) was introduced to apply the traditional cooperative refinements.

Our methodology is based on the global games theory, where the equilibrium selection is obtained through perturbations by allowing some arbitrarily small uncertainty in the payoff structure. Interestingly, the equilibrium selected with our stability concept was not only different, but also conflicts with those predicted by the traditional cooperative refinements. As a consequence, a first insight of this paper was to show that the equilibria selected under the cooperative notions of stability are not robust to the incomplete information structure considered.

In Proposition 3 we showed that the set of strategy profiles leading to efficient allocations is contained in the set of stable equilibria when the stability notions are cooperative. In other words, in our link formation game when the payoff functions belong to our class of supermodular functions, we do not have a conflict between stability and efficiency when cooperative refinements are used. On the contrary, from the Corollary of Proposition 2, we found that the conflict appears when our equilibrium selection technique is used.

>From an applied point of view, the paper highlights the importance of two standard assumptions in the link formation literature. First, the assumption of complete information can be the origin of the multiplicity of networks supported by Nash Equilibria in link formation games. This multiplicity disappears in our environment under

incomplete information because, from Proposition 2, there is a unique strategy profile that survives the iterative elimination of strictly dominated strategies and so any additional refinement is meaningless. Second, the possibility of cooperation among coalitions of agents seems to be a strong assumption in a link formation game, specially in the presence of coordination problems. This observation, and the conflict between the equilibria selected under a cooperative and a global games approach, raise some doubts about which criteria is satisfied by the forming networks in reality.

In the three player example discussed in section 2, in the interval $\frac{c}{1+\alpha} < x < \frac{4c}{2+4\alpha+\beta}$, all the cooperative refinements predict the formation of the complete network; however, our approach predicts the formation of the empty network. In particular, pairwise stability implies that a pair of agents can be strictly better off if they cooperate, however the strategies required to support this behavior do not survive the iterated elimination of strictly dominated strategies under any level of incomplete information in the parameter $x$.

The conflict between efficiency and stability in networks under the global games approach does not come as a surprise.[25] In fact Carlson and van Damme (1996) considered a 2×2 coordination game where the risk dominant equilibrium selected is not necessarily efficient. In this paper we extend the same kind of conclusion to a link formation game with $N+1$ players. On the other hand, the conflict between the equilibria selected under the cooperative refinements and the Global Games approach is originated in the use of different, and sometimes conflicting, stability concepts. The question here is which one seems more reasonable for link formation games. The use of cooperative refinements in games with strategic complementarities in the presence of a coordination problem is equivalent to ignoring the coordination problem itself. Our non cooperative approach then constitutes a feasible alternative.

In terms of future research, there are several important issues that remains open.

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[25] We thank to both anonymous referees for pointed out this issue.

For example, the class of fuctions where the global games approach can be applied is still narrow, especially because of assumption 1. We are studying how to relax this assumption in order to make the key results of the paper available to a bigger class of payoff functions. The role of symmetry in the payoff structure under complete information should also be analyzed, especially when the parameter of benefits becomes agent specific. Finally, we should also study the effect of introducing a dynamic dimension to the model. By doing this, we will be able to study entry and exit decisions in a market characterized by network interactions.

## 6 Appendix

**Definition**

*Pairwise Stable Nash Equilibria* ($PS$)

The concept of pairwise stability is due to Jackson and Wolinsky (1996) and it is directly defined over the networks, independently of the link formation process. It says that the network will be pairwise stable when each pair of agents do not have incentives to add or sever a link. It is clear from the definition of our game that adding a link requires both parties to agree, but any agent can sever a link unilaterally.

In our game, however, we need to use a variant of the original definition which depends directly over strategies (rather than networks).

Formally, let $G = (\mathcal{N}, \{A_i\}_{i\in\mathcal{N}}, \{\pi_i\}_{i\in\mathcal{N}})$ denote our link formation game in strategic form (defined in section 4.1). Define $E_{ij}$ as a $N \times (N+1)$ matrix of zeros, except the $ij$ element which is 1. We say that a strategy profile $a^* \in A = \underset{i=1}{\overset{N+1}{\times}} A_i$ is a *Pairwise Stable Equilibrium* ($PS$) of the game $G$ iff $a^* \in NE(G)$ and:

(i) If the link $ij$ exists then players $i$ and $j$ do not benefit if the link is broken. Formally, for all $i$, $j$ such that $[a^*]_{ij}[a^*]_{ji} = 1$,[26] we have:

[26] The notation $[a^*]_{ij}$ refers to the $ij$ element of the matrix $a^*$. In other words, it refers to the link

$\pi_i(a^*) \geq Max\{\pi_i(a^* - E_{ij}), \pi_i(a^* - E_{ij} - E_{ji})\}$ and,

$\pi_j(a^*) \geq Max\{\pi_j(a^* - E_{ji}), \pi_j(a^* - E_{ji} - E_{ij})\}$

(ii) If the link $ij$ does not exist then if one player benefits from the existence of the link then the other is damaged. Formally, for all $i$, $j$ such that $[a^*]_{ij}[a^*]_{ji} = 0$,

if $\pi_i(a^*) < \pi_i(a^* + E_{ij}(1 - [a^*]_{ij}) + E_{ji}(1 - [a^*]_{ji}))$

then $\pi_j(a^*) > \pi_j(a^* + E_{ij}(1 - [a^*]_{ij}) + E_{ji}(1 - [a^*]_{ji}))$

The connection between the two definitions of pairwise stability is direct: the set of Pairwise Stable Nash Equilibria of the game, $PS(\Gamma)$, corresponds to the subset of Nash equilibria leading to Pairwise Stable networks.

**Proof of Lemma 1**:

Let us first prove the lemma for $\underline{x}$. Consider $a_i$, $a_i^* \in A_i \backslash \mathbf{0}$ and $x \in [\underline{X}, \overline{X}]$. From A4 and $l = I - 1$ we know that

$\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) = \lambda(a_i)\Delta\pi(a_i', \mathbf{0}; a_{-i}, x) \quad \forall a_i' \in A_i^1$

$\Delta\pi(a_i^*, \mathbf{0}; a_{-i} = [\mathbf{1}], x) = \lambda(a_i^*)\Delta\pi(a_i', \mathbf{0}; a_{-i}, x) \quad \forall a_i' \in A_i^1$

which implies $\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) = \frac{\lambda(a_i)}{\lambda(a_i^*)}\Delta\pi(a_i^*, \mathbf{0}; a_{-i} = [\mathbf{1}], x)$

where by A4 $\lambda(a_i) \neq 0$ and $\lambda(a_i^*) \neq 0$. Thus, if $\underline{x}$ satisfies $\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$ then $\Delta\pi(a_i^*, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$.

Let us now prove for $\overline{x}$. Consider $a_i$, $a_i^* \in A_i \backslash \mathbf{1}$ and $x \in [\underline{X}, \overline{X}]$. In general we can write:

$\Delta\pi(\mathbf{1}, a_i; a_{-i}, x) = \Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, x) - \Delta\pi(a_i, \mathbf{0}; a_{-i}, x)$

from A4 and $l = 0$

$\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, x) = \lambda(\mathbf{1})\Delta\pi(a_i', \mathbf{0}; [\mathbf{0}], x) \quad \forall a_i' \in A_i^1$

$\Delta\pi(a_i, \mathbf{0}; [\mathbf{0}], x) = \lambda(a_i)\Delta\pi(a_i', \mathbf{0}; [\mathbf{0}], x) \quad \forall a_i' \in A_i^1$

then

intention of player $j$ with player $i$.

$\Delta\pi(\mathbf{1}, a_i; a_{-i} = [\mathbf{1}], x) = (\lambda(\mathbf{1}) - \lambda(a_i))\Delta\pi(a_i', \mathbf{0}; [\mathbf{0}], x) \qquad \forall a_i' \in A_i^1$ and analogously

$\Delta\pi(\mathbf{1}, a_i^*; a_{-i} = [\mathbf{1}], x) = (\lambda(\mathbf{1}) - \lambda(a_i^*))\Delta\pi(a_i', \mathbf{0}; [\mathbf{0}], x) \qquad \forall a_i' \in A_i^1$

Since by A4 $(\lambda(\mathbf{1}) - \lambda(a_i)) \neq 0$ and $(\lambda(\mathbf{1}) - \lambda(a_i^*)) \neq 0$ we conclude as above.

**Proof of Proposition 2**

Denoting $S_i^t$ the player $i$'s set of strategies that survives $t$ rounds of deletion of interim strictly dominated strategies, the process of iterated elimination is defined recursively as follows: set $S_i^0 \equiv S_i$ and for all $t > 0$

$$S_i^t \equiv \left\{ \begin{array}{c} s_i \in S_i^{t-1} : \nexists s_i' \in S_i^{t-1} \; s.t. \; \Pi(s_i'(x_i), s_{-i}, x_i \mid x_i) \geq \Pi(s_i(x_i), s_{-i}, x_i \mid x_i) \; \forall x_i \\ and \; with \; strict \; inequality \; for \; some \, x_i, \; \forall s_{-i} \in S_{-i}^{t-1} \end{array} \right\}$$

Consider a link formation game $G(\sigma)$. Under assumptions A1 to A6, we will argue by induction that set $S_i^t$ satisfies:

$S_i^t = \{s_i : s_i(x_i) = \mathbf{0} \; if \; x_i < \underline{x}^t \; and \; s_i(x_i) = \mathbf{1} \; if \; x_i > \overline{x}^t\}$,

where $\underline{x}^t$ and $\overline{x}^t$ are defined recursively as

$\overline{x}^t = \max\{x : \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \overline{x}^{t-1}))_{j \neq i}, x) = 0\}$

$\underline{x}^t = \min\{x : \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \underline{x}^{t-1}))_{j \neq i}, x) = 0\}$

The first round of elimination is described in the following lemma.

*Lemma 3: For all* $i \; \exists \; \underline{x}^1 > \underline{x}$ *and* $\overline{x}^1 < \overline{x}$ *s.t.*

$s_i \in S_i^1 \; iff \; s_i(x_i) = \{\mathbf{0} \; if \; x_i < \underline{x}^1 \; and \; \mathbf{1} \; if \; x_i > \overline{x}^1\}$

*where*

$$\overline{x}^1 = \max\{x : \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \overline{x}))_{j \neq i}, x) = 0\}$$

$$\underline{x}^1 = \min\{x : \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \underline{x}))_{j \neq i}, x) = 0\}$$

*proof*. Starting from the left: Player $i$ (henceforth P$i$) observes $x_i = \underline{x}$, from A1 (ID), if $s_i$ is a best response to a profile where every player is choosing a switching strategy $s_j(\cdot;\underline{x})$ $\forall j \neq i$, it will be a best response to any $s_{-i} \in S^0_{-i}$. Then player $i$' expected payoff difference between choosing action $a_i$ rather than action $\mathbf{0}$ can be written as

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j;\underline{x}))_{j\neq i}, \underline{x}) = \int_{x_{-i}} \Delta\pi(a_i, \mathbf{0}; s_j(x_j;\underline{x}))_{j\neq i}, \underline{x}) dF_{\sigma(-i)}(x_{-i} \mid x_i)$$

or equivalently by

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j;\underline{x}))_{j\neq i}, \underline{x}) = \sum_{a_{-i}\in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, \underline{x}) \Pr(a_{-i} \mid (s_j(x_j;\underline{x}))_{j\neq i}, \underline{x})$$

where in general $\Pr(a_{-i} \mid (s_{-i}, x)$ represent player $i$' beliefs about the action profile $a_{-i}$ conditional on other players' strategy $s_{-i}$.

Now, since, $\forall \sigma > 0$, $\forall a_{-i} \in A^h_{-i}$, $\Pr(a_{-i} \mid (s_j(x_j;\underline{x}))_{j\neq i}, \underline{x}) = \frac{1}{2^{I-1}} > 0$, then

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j;\underline{x}))_{j\neq i}, \underline{x}) = \frac{1}{2^{I-1}} \sum_{a_{-i}\in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, \underline{x})$$

By assumptions A1 (ID) and A5 (IV) $\forall a_i \in A_i$, $\forall a_{-i} \in A^h_{-i}$ $\Delta\pi(a_i, \mathbf{0}; a_{-i}, \underline{x}) \leq 0$. By assumption A1 (ID) part a, at least one element is strictly negative, then $\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j;\underline{x}))_{j\neq i}, \underline{x}) < 0$. Therefore P$i$, upon observing $x_i = \underline{x}$, will play action $a_i = \mathbf{0}$.

Now, if P$i$ receive a value $x_i = \underline{x} + \sigma$

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j;\underline{x}))_{j\neq i}, \underline{x} + \sigma) = \Delta\pi(a_i, \mathbf{0}; s_{-i} = [\mathbf{1}], x_i = \underline{x} + \sigma)$$

By assumption A3 (M) $\Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], \underline{x} + \sigma) > \Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x_i = \underline{x})$, and by assumptions A5 (IV) $\Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x_i = \underline{x}) = 0$. Then $\Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x_i = \underline{x} + \sigma) > 0$.

Given continuity of the expected utility function and using the *intermediate value theorem:*

$\forall a_i \neq \mathbf{0}$ and $\forall \sigma > 0, \ \exists\, \underline{x}^1$ s.t $\ \underline{x} < \underline{x}^1 < \underline{x} + \sigma,$ where $\underline{x}^1 = \min \{x \mid \text{equation (6) holds}\}$

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j; \underline{x}))_{j \neq i}, x) = \sum_{a_{-i} \in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) \Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j \neq i}, x) = 0 \tag{6}$$

Following an equivalent argument of Lemma 1, let us now prove that any $x$ such equation (6) holds is independent of $a_i$ (in particular it will be true for $\underline{x}^1$) :

Since $\sum_{a_{-i} \in A^h_{-i}} = \sum_{l=0}^{I-1} \sum_{a_{-i} \in A^{h,l}_{-i}}$. we can re-write equation 6 as

$\sum_{l=0}^{I-1} \sum_{a_{-i} \in A^{h,l}_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) \Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j \neq i}, x) = 0$

but $\Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j \neq i}, x)$ just depend on $l$ so

$\sum_{l=0}^{I-1} \Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j \neq i}, x) \sum_{a_{-i} \in A^{h,l}_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) = 0.$

$>$From A4 we know that

$\sum_{a_{-i} \in A^{h,l}_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) = \lambda(a_i) \sum_{a_{-i} \in A^{h,l}_{-i}} \Delta\pi(a'_i, \mathbf{0}; a_{-i}, x)\ \forall l = 0, ..., I-1\ \ \forall a'_i \in A^1_i$

then

$\sum_{l=0}^{I-1} \Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j \neq i}, x)[\lambda(a_i) \sum_{a_{-i} \in A^{h,l}_{-i}} \Delta\pi(a'_i, \mathbf{0}; a_{-i}, x)] = 0$

which can be written as

$\lambda(a_i)[\sum_{l=0}^{I-1} \Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j \neq i}, x) \sum_{a_{-i} \in A^{h,l}_{-i}} \Delta\pi(a'_i, \mathbf{0}; a_{-i}, x)] = 0$

then any $x$ such equation (6) holds is independent of $a_i$.

Now we know that $\underline{x}^1$ is independent of $a_i$. Then in particular if $a_i = \mathbf{1}$

$$\underline{x}^1 = \min \{x \mid \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \underline{x}))_{j \neq i}, x) = 0\}$$

Starting from the right and using an equivalent argument we conclude that:

$\forall a_i \neq \mathbf{1}$ and $\forall \sigma > 0$, $\exists\, \overline{x}^1$ s.t $\overline{x} > \overline{x}^1 > \overline{x} - \sigma$, where $\overline{x}^1 = \max\{x \mid \text{equation (7) holds}\}$

$$\Delta\Pi(\mathbf{1}, a_i; (s_j(x_j; \overline{x}))_{j\neq i}, x) = \sum_{a_{-i} \in A^h_{-i}} \Delta\pi(\mathbf{1}, a_i; a_{-i}, x) \Pr(a_{-i} \mid (s_j(x_j; \overline{x}))_{j\neq i}, x) = 0 \tag{7}$$

Since $\overline{x}^1$ is independent of $a_i$, then in particular if $a_i = \mathbf{0}$

$$\overline{x}^1 = \max\{x \mid \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \overline{x}))_{j\neq i}, x) = 0\} \blacksquare_{Lemma\ 3}$$

Repeating the process described in lemma 3, it is easy to prove by induction that $\exists\, \underline{x}^t > \underline{x}^{t-1}$ *and* $\overline{x}^t < \overline{x}^{t-1}$ *s.t.*

$S_i^t = \{s_i : \ s_i(x_i) = \mathbf{0} \ if \ x_i < \underline{x}^t \ and \ s_i(x_i) = \mathbf{1} \ if \ x_i > \overline{x}^t\}$

*where*

$$\overline{x}^t = \max\left\{x : \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \overline{x}^{t-1}))_{j\neq i}, x) = 0\right\}$$

$$\underline{x}^t = \min\left\{x : \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \underline{x}^{t-1}))_{j\neq i}, x) = 0\right\}$$

This process generates an increasing sequence $\{\underline{x}^t\}$ and a decreasing sequence $\{\overline{x}^t\}$. Let us suppose there exists limit points $\underline{x}^\infty$ and $\overline{x}^\infty$, then from equation (6) $\forall a_i$

$$\sum_{a_{-i} \in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, \underline{x}^\infty) \Pr(a_{-i} \mid (s_j(x_j; \underline{x}^\infty))_{j\neq i}, \underline{x}^\infty) = 0$$

Since $\Pr(a_{-i} \mid (s_j(x_j; \underline{x}^\infty))_{j\neq i}, \underline{x}^\infty) = \frac{1}{2^{I-1}}$, then

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j; \underline{x}^\infty))_{j\neq i}, \underline{x}^\infty) = \frac{1}{2^{I-1}} \sum_{a_{-i} \in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, \underline{x}^\infty) = 0$$

By assumption A5 (IV), in particular, this is true for $a_i = \mathbf{1}$, then

$$\sum_{a_{-i} \in A^h_{-i}} \Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, \underline{x}^{\infty}) = 0 \tag{8}$$

Equivalently from equation (7), for the limit point $\overline{x}^{\infty}$ we get

$$\Delta\Pi(\mathbf{1}, a_i; (s_j(x_j; \overline{x}^{\infty}))_{j \neq i}, \overline{x}^{\infty}) = \frac{1}{2^{I-1}} \sum_{a_{-i} \in A^h_{-i}} \Delta\pi(\mathbf{1}, a_i; a_{-i}, \overline{x}^{\infty}) = 0$$

By assumption A5 (IV) this is in particular true for $a_i = \mathbf{0}$, then

$$\sum_{a_{-i} \in A^h_{-i}} \Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, \overline{x}^{\infty}) = 0 \tag{9}$$

Finally, it is easy to see that equations (8) and (9) are the same, and from assumption A6 (SC) $\underline{x}^{\infty} = \overline{x}^{\infty} = k^*$. Then $S^{\infty} = \bigcap_{t=0}^{\infty} S^t = \left\{ (s_i(x_i; k^*))_{i=1}^{I} \right\}$ ■

**Proof of Proposition 3**

(b) First we have to prove that $a = [\mathbf{0}]$ and $a = [\mathbf{1}]$ are indeed Nash Equilibria of the game when $x < \underline{x}$ and $x > \underline{x}$ respectively.

Consider first $x > \underline{x}$ and $a_i \neq \mathbf{0}$, $a_i \neq \mathbf{1}$. By A3 (M) we have:

$\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) - \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) > c \left\| a_i - \mathbf{0} \right\| (x - \underline{x}) > 0$

and by A5 (IV) $\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$, so $\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) > 0$.

On the other hand, by A4 (PID) we have:

$\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i} = [\mathbf{1}], x) = \frac{\lambda(\mathbf{1})}{\lambda(a_i)} \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) > \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x)$

Finally, using A7 (SQP), and the fact that the following inequality trivially holds for $a_i = \mathbf{0}$, we have:

$$\pi(\mathbf{1}; a_{-i} = [\mathbf{1}], x) > \pi(a_i; a_{-i} = [\mathbf{1}], x) \quad \forall a_i \neq \mathbf{1}, x > \underline{x}. \tag{10}$$

In other words, when the others are playing $a_{-i} = [\mathbf{1}]$, then play $a_i = \mathbf{1}$ is a strict best response. As a result, $a = [\mathbf{1}]$ is a strict $NE$ of the game when $x > \underline{x}$.

Now we are going to prove that $a = [\mathbf{1}]$ is a $SNE$ of the game when $x > \underline{x}$ and then, by relations in (2), it is an stable equilibria under all the cooperative refinements.

We are going to prove that:

$\pi(\mathbf{1}; [\mathbf{1}], x) \geq \pi(a_i; a_{-i}, x) \quad \forall\ x > \underline{x}, \forall a \in A,\ a \neq [\mathbf{1}]$

which is a condition that implies that the strategy profile $a = [\mathbf{1}]$ is a Strong Nash Equilibrium (SNE).

Consider any $a \in A,\ a \neq [\mathbf{1}]$ and any $x > \underline{x}$. By A1 (ID) and A7 (SQP) we have:

$\pi(a_i; a_{-i}, x) = \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) \leq \Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x) = \pi(a_i; [\mathbf{1}], x)$

and using equation (10) we have:

$\pi(a_i; [\mathbf{1}], x) \leq \pi(\mathbf{1}; [\mathbf{1}], x)$

which completes the proof that $a = [\mathbf{1}]$ is a $SNE$ of the game when $x > \underline{x}$.

Now we analyze the case $x < \underline{x}$. In this case, the strategy $a_i = \mathbf{0}$ is a strictly dominant strategy for player $i$ because $\forall a_i \neq \mathbf{0}$ and $\forall a_{-i} \in A_{-i}$, by A1 (ID), A3 (M) and A5 (IV) we have:

$\Delta\pi(a_i, \mathbf{0}; a_{-i}, x) \leq \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) < \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$

and using A7 (SQP):

$\pi(a_i; a_{-i}, x) < 0 = \pi(\mathbf{0}; a_{-i}, x)$

In particular, considering $a_{-i} = [\mathbf{0}]$, we obtain that $a = [\mathbf{0}]$ is a strict $NE$ of the game when $x < \underline{x}$.

Moreover, given any strategy profile $a \neq [\mathbf{0}]$ (not necessarily a Nash equilibrium) and any $x < \underline{x}$ we have:

$\pi(\mathbf{0}; a_{-i} = [\mathbf{0}], x) \geq \pi(a_i; a_{-i}, x)$ and then $a = [\mathbf{0}]$ is a $SNE$ of the game.

Finally, when $x = \underline{x}$, the strategy profiles $a = [\mathbf{0}]$ and $a = [\mathbf{1}]$ lead to a payoff zero (by assumptions A7 (SQP) and A5 (IV) respectively), and using A1 (ID) and A5 (IV), for any strategy profile $a \in A$:

$\pi(a_i; a_{-i}, \underline{x}) \leq \pi(\mathbf{1}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$

As a consequence there is no profitable deviation for player $i$ from $a = [\mathbf{0}]$ or $a = [\mathbf{1}]$, so these profiles are Nash Equilibria. Using the same assumptions, there is no other profile where all the players in a coalition can obtain a positive payoff and, consequently, these profiles are also Strong Nash Equilibria. Moreover, if there exists any other efficient strategy profile under $x = \underline{x}$, the payoff for any player $i$ would be zero and then, it would also be a $SNE$ of the game.

(a) By definition, the set of Efficient Allocations of the game $G$ is given by the strategy profiles that solves:

$$\max_{a \in A} \sum_{i=1}^{N+1} \pi_i(a_i; a_{-i}, x)$$

$>$From the proof of part (b), we know that, when $x > \underline{x}$ the strategy profile $a = [\mathbf{1}]$ is a Nash equilibrium satisfying:

$\pi_i(\mathbf{1}; [\mathbf{1}], x) \geq \pi_i(a_i; a_{-i}, x) \quad \forall\ x > \underline{x}\ ,\ \ a \neq [\mathbf{1}],\ \ i = 1...N+1$

But if the strategy profile $a \neq [\mathbf{1}]$ then there exists $j$ so that $a_j \neq \mathbf{1}$ and for this agent, using A7 (SQP), A1 (ID) and A4 (PID) we have:

$\pi_j(a_j; a_{-j}, x) \leq \pi_j(a_j; [\mathbf{1}], x) < \pi_j(\mathbf{1}; [\mathbf{1}], x) \quad \forall\ x > \underline{x}$

and then the unique Efficient Allocation when $x > \underline{x}$ is given by the strategy profile $a = [\mathbf{1}]$.

An analogous argument leads us to prove that the unique Efficient Allocation when $x < \underline{x}$ is given by the strategy profile $a = [\mathbf{0}]$.■

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